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# On two conjectures about the proper connection number of graphs[☆]

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ABSTRACT

A path in an edge-colored graph is called proper if no two consecutive edges of the path receive the same color. For a connected graph $G$, the proper connection number $pc(G)$ of $G$ is defined as the minimum number of colors needed to color its edges so that every pair of distinct vertices of $G$ is connected by at least one proper path in $G$. In this paper, we consider two conjectures on the proper connection number of graphs. The first conjecture states that if $G$ is a noncomplete graph with connectivity $\kappa(G) = 2$ and minimum degree $\delta(G) \geq 3$, then $pc(G) = 2$, posed by Borozan et al. (2012). We give a family of counterexamples to disprove this conjecture. However, from a result of Thomassen it follows that 3-edge-connected noncomplete graphs have proper connection number 2. Using this result, we can prove that if $G$ is a 2-connected noncomplete graph with $diam(G) = 3$, then $pc(G) = 2$, which solves the second conjecture we want to mention, posed by Li and Magnant (2015).


## 1. Introduction

All graphs in this paper are simple, finite and undirected. We follow [2] for graph theoretical notation and terminology not defined here. Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N(v)$ denote the set of neighbors of $v$. For a subset $U \subseteq V(G)$, let $N(U) = \left(\bigcup_{v\in U} N(v)\right) \setminus U$. For any two disjoint subsets $X$ and $Y$ of $V(G)$, we use $E(X, Y)$ to denote the set of edges of $G$ that have one end in $X$ and the other in $Y$. Denote by $|E(X, Y)|$ the number of edges in $E(X, Y)$. An $(X, Y)$-path is a path which starts at a vertex of $X$, ends at a vertex of $Y$, and whose internal vertices belong to neither $X$ nor $Y$.

Let $G$ be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. If adjacent edges of $G$ are assigned different colors by $c$, then $c$ is called a *proper (edge-)coloring*. For a graph $G$, the minimum number of colors needed in a proper coloring of $G$ is referred to as the *edge-chromatic number* of $G$ and denoted by $\chi'(G)$. A path of an edge-colored graph $G$ is said to be a *rainbow path* if no two edges on the path have the same color. The graph $G$ is called *rainbow connected* if for any two vertices there is a rainbow path of $G$ connecting them. An edge-coloring of a connected graph is a *rainbow connecting coloring* if it makes the graph rainbow connected. For a connected graph $G$, the *rainbow connection number* $rc(G)$ of $G$ is defined to be the smallest number of colors that are needed in order

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to make $G$ rainbow connected. The concept of rainbow connection of graphs was introduced by Chartrand et al. [5] in 2008. Readers who are interested in this topic can see [11,12] for a survey.

Motivated by the rainbow coloring and proper coloring in graphs, Andrews et al. [1] and Borozan et al. [3] introduced the concept of proper-path coloring. Let $G$ be a nontrivial connected graph with an edge-coloring. A path in $G$ is called a *proper path* if no two adjacent edges of the path are colored with the same color. An edge-coloring of a connected graph $G$ is a *proper-path coloring* if every pair of distinct vertices of $G$ is connected by a proper path in $G$. If $k$ colors are used, then $c$ is referred to as a *proper-path k-coloring*. An edge-colored graph $G$ is called *proper connected* if every pair of distinct vertices of $G$ is connected by a proper path. For a connected graph $G$, the *proper connection number* of $G$, denoted by $pc(G)$, is defined as the smallest number of colors that are needed in order to make $G$ proper connected.

The proper connection of graphs has the following application background. When building a communication network between wireless signal towers, one fundamental requirement is that the network be connected. If there cannot be a direct connection between two towers $A$ and $B$, say for example if there is a mountain in between, there must be a route through other towers to get from $A$ to $B$. As a wireless transmission passes through a signal tower, to avoid interference, it would help if the incoming signal and the outgoing signal do not share the same frequency. Suppose that we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and assign a color to each edge based on the assigned frequency used for the communication. Then, the number of frequencies needed to assign frequencies to the connections between towers so that there is always a path avoiding interference between each pair of towers is precisely the proper connection number of the corresponding graph.

Let $G$ be a nontrivial connected graph of order $n$ (number of vertices) and size $m$ (number of edges). Then the proper connection number of $G$ has the following clear bounds:

$$1 \leq pc(G) \leq \min\{rc(G), \chi'(G)\} \leq m.$$

Furthermore, $pc(G) = 1$ if and only if $G = K_n$, $pc(G) = m$ if and only if $G = K_{1,m}$ is a star of size $m$.

Given an edge-colored path $P = v_1v_2 \ldots v_{s-1}v_s$ between any two vertices $v_1$ and $v_s$, we denote by $start(P)$ the color of the first edge in the path, i.e., $c(v_1v_2)$, and by $end(P)$ the color of the last edge in the path, i.e., $c(v_{s-1}v_s)$. If $P$ is just the edge $v_1v_s$, then $start(P) = end(P) = c(v_1v_s)$.

**Definition 1.1** (*[3]*)**.** Let $c$ be an edge-coloring of $G$ that makes $G$ proper connected. We say that $G$ has the *strong property under c* if for any pair of vertices $u, v \in V(G)$, there exist two proper paths $P_1, P_2$ connecting them (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.

Next we list the following four lemmas, which will be used in this work.

**Lemma 1.1** (*[1]*)**.** *If $G$ is a nontrivial connected graph and $H$ is a connected spanning subgraph of $G$, then $pc(G) \leq pc(H)$. In particular, $pc(G) \leq pc(T)$ for every spanning tree $T$ of $G$.*

In fact, Lemma 1.1 also states that the proper connection number is monotonic under adding edges.

**Lemma 1.2** (*[3]*)**.** *If $G$ is a 2-connected graph, then $pc(G) \leq 3$. Furthermore, there exists a 3-edge-coloring $c$ of $G$ such that $G$ has the strong property under $c$.*

**Lemma 1.3** (*[3,9]*)**.** *If $G$ is a connected bridgeless bipartite graph, then $pc(G) \leq 2$. Furthermore, there exists a 2-edge-coloring $c$ of $G$ such that $G$ has the strong property under $c$.*

**Lemma 1.4** (*[1]*)**.** *Let $G$ be a connected graph and $v$ a vertex not in $G$. If $pc(G) = 2$, then $pc(G \cup v) = 2$ as long as $d(v) \geq 2$, that is, we connect $v$ to $G$ by using at least two edges.*

For more details we refer to [1,3,6–8,13] and a dynamic survey [10].

The first conjecture we will consider in this paper is as follows, which was posed by Borozan et al. in [3].

**Conjecture 1.1** (*[3]*)**.** *If $\kappa(G) = 2$ and $\delta(G) \geq 3$, then $pc(G) = 2$.*

As observed in the example of [[3], Proposition 3], in which the graph $G$ satisfies that $\kappa(G) = 2$ and $\delta(G) = 2$ but $pc(G) = 3$, the bound on $\delta(G)$ in Conjecture 1.1 would be sharp if the conjecture is true.

The second conjecture we will consider is as follows, which was posed by Li and Magnant in [10].

**Conjecture 1.2** (*[10]*)**.** *If $G$ is a 2-connected noncomplete graph with $diam(G) = 3$, then $pc(G) = 2$.*

In Section 2 we disprove Conjecture 1.1. In Section 3 we prove Conjecture 1.2.