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On light neighborhoods of 5-vertices in 3-polytopes with minimum degree 5 [☆]

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ABSTRACT

Let w_Δ be the minimum integer W with the property that every 3-polytope with minimum degree 5 and maximum degree Δ has a vertex of degree 5 with the degree-sum (weight) of all vertices in its closed neighborhood at most W.

Trivially, $w_5 = 30$ and $w_6 = 35$. In 1940, Lebesgue proved $w_\Delta \le \Delta + 31$ for all $\Delta \ge 5$ and $w_\Delta \le \Delta + 27$ for $\Delta \ge 41$.

In 1998, the first Lebesgue's result was improved by Borodin and Woodall to $w_\Delta \le \Delta + 30$. This bound is sharp for $\Delta = 7$ due to Borodin (1992) and Jendrol' and Madaras (1996), $\Delta = 9$ due to Borodin and Ivanova (2013), $\Delta = 10$ due to Jendrol' and Madaras (1996), and $\Delta = 12$ due to Borodin and Woodall (1998). As for the second Lebesgue's bound, Borodin et al. (2014) proved that $w_\Delta \le \Delta + 27$ for $\Delta \ge 28$, but $w_{20} \ge 48$; the former fact was extended by Borodin and Ivanova (2016) to $w_\Delta \le \Delta + 27$ for $\Delta \ge 23$.

The purpose of this paper is to prove $w_\Delta \le \Delta + 29$ whenever $\Delta \ge 13$ and show that $w_8 = 38$, $w_{11} = 41$, and $w_{13} = 42$. Thus w_Δ remains unknown only for $14 \le \Delta \le 22$.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a plane graph G is the number of its incident edges. A *k-vertex* (*k-neighbor*, *k-face*) is a vertex (adjacent vertex, face) with degree k, a k^+*-vertex* has degree at least k, etc. The maximum and minimum vertex degrees of G are $\Delta(G)$ and $\delta(G)$, respectively. We will drop the arguments whenever this does not lead to confusion.

As proved by Steinitz [9], the 3-connected plane graphs are planar representations of the convex 3-dimensional polytopes, called hereafter 3-polytopes.

In this note, we consider the class $\mathbf{P_5}$ of 3-polytopes with $\delta = 5$. An S_k stands for a k-star with k rays, $k \ge 1$, centered at a 5-vertex.

In 1904, Wernicke [10] proved that if $P_5 \in \mathbf{P_5}$ then P_5 contains a vertex of degree 5 adjacent to a 6^--vertex. This result was strengthened by Franklin [6] in 1922 to the existence of a 5-vertex with two 6^--neighbors. In 1940, Lebesgue [8, p. 36] gave an approximate description of the neighborhoods of vertices of degree 5, that is of S_5's, in $\mathbf{P_5}$.

By $w(S_k)$, where $1 \le k \le 5$, we mean the maximum over all $P_5 \in \mathbf{P_5}$ of the minimum degree-sum (weight) of the vertices of S_ks in P_5's.

The bounds $w(S_1) \le 11$ (Wernicke [10]) and $w(S_2) \le 17$ (Franklin [6]) are tight. It was proved by Lebesgue [8] that $w(S_3) \le 24$ and $w(S_4) \le 31$, which was improved much later to the following tight bounds: $w(S_3) \le 23$

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(Jendrol'–Madaras [7]) and $w(S_4) \le 30$ (Borodin–Woodall [5]). Note that $w(S_3) \le 23$ easily implies $w(S_2) \le 17$ and immediately follows from $w(S_4) \le 30$ (it suffices to delete a vertex of maximum degree from a star of minimum weight).

Jendrol' and Madaras [7] showed that $w(S_5) = \infty$ by constructing a $P_5 \in \mathbf{P_5}$ with each 5-vertex adjacent to a $\Delta(P_5)$-vertex, where $\Delta(P_5)$ is arbitrarily large. In studying light 5-stars, this fact leads to the following definition.

Let w_Δ be the minimum integer W with the property that every P_5 in $\mathbf{P_5}$ with $\Delta(P_5) = \Delta$ has an S_5 with weight at most W.

Trivially, $w_5 = 30$ and $w_6 = 35$. It follows from Lebesgue [8] that $w_\Delta \le \Delta + 31$ for all $\Delta \ge 5$ and $w_\Delta \le \Delta + 27$ for $\Delta \ge 41$.

The above mentioned result $w(S_4) \le 30$ in Borodin–Woodall [5] implies $w_\Delta \le \Delta + 30$. This bound is sharp for $\Delta = 7$ due to Borodin [1] and Jendrol'–Madaras [7], for $\Delta = 9$, as shown in Borodin–Ivanova [2], $\Delta = 10$ due to [7], and $\Delta = 12$, as shown in [5].

As for Lebesgue's bound $w_\Delta \le \Delta+27$ for $\Delta \ge 41$, it was strengthened by Borodin, Ivanova, and Jensen [4] to $w_\Delta \le \Delta+27$ for $\Delta \ge 28$. On the other hand, $w_{20} \ge 48$, as shown in [4]. Recently, we proved [3] that $w_\Delta \le \Delta + 27$ for $\Delta \ge 23$.

The purpose of our paper is to prove that $w_\Delta \le \Delta + 29$ whenever $\Delta \ge 13$ and show that $w_8 = 38$, $w_{11} = 41$, and $w_{13} = 42$.

Theorem 1. *For every integer Δ such that $\Delta \ge 13$, every 3-polytope P_5 with $\delta(P_5) = 5$ and $\Delta(P_5) = \Delta$ has a 5-star with central 5-vertex which has weight at most $\Delta + 29$.*

Proposition 2. *There exist 3-polytopes confirming that $w_8 = 38$, $w_{11} = 41$, and $w_{13} = 42$.*

A natural problem for further research is to find w_Δ for the remaining values $14 \le \Delta \le 22$.

2. Proving Proposition 2

For the cases $w_8 = 38$ and $w_{11} = 41$, see Figs. 1 and 2, respectively. To prove $w_{13} = 42$, we take the graph in Fig. 3 and add a 12-vertex in its exterior face. As a result, we have a triangulation with $\delta = 5$, $\Delta = 13$, and $w_{13} = 4 \times 5 + 10 + 12 = \Delta + 29$, as desired.

3. Proving Theorem 1

Suppose G' is a counterexample to Theorem 1. Let G be a maximal counterexample such that $V(G') = V(G)$ and $E(G') \subseteq E(G)$. Clearly G is 3-connected, since G is a counterexample to Theorem 1. Denote the sets of vertices, edges, and faces of G by V, E and F, respectively. Euler's formula $|V| - |E| + |F| = 2$ for G yields

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(x)$ to x whenever $x \in V \cup F$ as follows: $\mu(v) = d(v) - 6$ if $v \in V$ and $\mu(f) = 2d(f) - 6$ if $f \in F$. Note that only 5-vertices have a negative initial charge.

Using the properties of G as a counterexample to Theorem 1, we will define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative whenever $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12, and this contradiction will complete the proof Theorem 1.

3.1. Structural properties of G

In what follows, we will need the simple structural properties of G expressed by (SP1) and (SP2).

(SP1) *The boundary $\partial(f)$ of every face of G is an induced cycle; that is, $\partial(f)$ is a cycle, and no two nonconsecutive vertices of $\partial(f)$ are adjacent.*

This follows from the planarity and 3-connectedness of G.

(SP2) *No 4^+-face can be incident with more than two 5-vertices.*

Otherwise, there are 5-vertices v, w that are not consecutive along $\partial(f)$, and then $G + vw$ is also a counterexample to Theorem 1, which contradicts the maximality of G. Let $v_1, \ldots, v_{d(v)}$ denote the neighbors of a vertex v in cyclic order round v. The vertex v is *simplicial* if all its incident faces are 3-faces.

If v_i is a simplicial 5-vertex, then v_i is a *strong*, *semiweak* or *weak* neighbor of (a not necessarily simplicial vertex) v according as both, one or none of v_{i-1}, v_{i+1} are 6^+-vertices, and v_i is a *twice-weak* neighbor of v if v_j is a simplicial 5-vertex whenever $|j - i| \le 2$ (modulo $d(v)$), see Fig. 4.

A (simplicial) 5-vertex v is *bad* if v is a twice weak neighbor of a 10-vertex v_2 and a weak neighbor for a 13^+-vertex v_5. If v is bad, then its 5-neighbor v_4 that lies in common 3-faces with its 5-neighbor v_3 and 13^+-neighbor v_5 is *good for* v_5. These notions are illustrated in Fig. 5. By $w(v)$ we mean the degree-sum of the closed neighborhood of v.