Discrete Mathematics 340 (2017) 2243–2249

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Some more uniformly resolvable designs with block sizes 2 and 4

Hengjia Wei [a], Gennian Ge [a,b,*]

[a] School of Mathematical Sciences, Capital Normal University, Beijing, 100048, China
[b] Beijing Center for Mathematics and Information Interdisciplinary Sciences, Beijing, 100048, China

ARTICLE INFO

Article history:
Received 11 September 2016
Received in revised form 24 April 2017
Accepted 26 April 2017

Keywords:
Uniformly resolvable design
Resolvable group divisible design
Frame

ABSTRACT

A uniformly resolvable design (URD) is a resolvable design in which each parallel class contains blocks of only one block size k. Such a class is denoted k-pc and for a given k the number of k-pcs is denoted r_k. The number of points of the URD is denoted by v. In the literature, the existence of URDs of block sizes k_1 and k_2 with $\{k_1, k_2\} \in \{\{2, 3\}, \{2, 4\}, \{3, 4\}\}$ has been studied with much efforts. For $\{k_1, k_2\} = \{2, 3\}$ or $\{3, 4\}$, the existence spectra have been determined completely, while for $\{k_1, k_2\} = \{2, 4\}$ there are still 1338 undetermined pairs of (v, r_2).

In this paper we continue the study on the existence of URDs of block sizes 2 and 4. We improve the known results by reducing the number of open cases from 1338 to 16. In addition, we prove that there exist 4-RGDDs of types 2^{178}, 2^{250} and 2^{334}.

1. Introduction

Let v and λ be positive integers, and let K and M be two sets of positive integers. A *group divisible design*, denoted GDD$(K, M; v)$, is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of v *points*, $\mathcal{G}$ is a partition of X into *groups*, and $\mathcal{B}$ is a collection of subsets of X, called *blocks*, such that

1. $|B| \in K$ for each $B \in \mathcal{B}$,
2. $|G| \in M$ for each $G \in \mathcal{G}$,
3. $|B \cap G| \le 1$ for each $B \in \mathcal{B}$ and each $G \in \mathcal{G}$, and
4. each pair of elements of X from distinct groups is contained in exactly one block.

If $K = \{k\}$, respectively $M = \{m\}$, then the GDD$(K, M; v)$ is simply denoted GDD$(k, M; v)$, respectively GDD$(K, m; v)$. A GDD$(K, 1; v)$ is called a *pairwise balanced design* and denoted PBD$(K; v)$. A GDD$(k, m; mk)$ is called a *transversal design* and denoted TD(k, m). We usually use an "exponential" notation to describe the multiset M: a K-GDD of type $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ is a GDD in which every block has size from the set K and in which there are u_i groups of size g_i, $i = 1, 2, \dots, s$.

In a GDD$(K, M; v)$ $(X, \mathcal{G}, \mathcal{B})$ a *parallel class* is a set of blocks, which partitions X. If $\mathcal{B}$ can be partitioned into parallel classes, then the GDD$(K, M; v)$ is said to be *resolvable* and denoted RGDD$(K, M; v)$. Analogously, a resolvable PBD$(K; v)$ is denoted RPBD$(K; v)$. A parallel class is said to be *uniform* if it contains blocks of only one size k (k-pc). If all parallel classes of an RPBD$(K; v)$ are uniform, the design is said to be *uniformly resolvable*. Here, a uniformly resolvable design RPBD$(K; v)$ is denoted URD$(K; v)$. In a URD$(K; v)$ the number of parallel classes with blocks of size k is denoted r_k, $k \in K$.

* Corresponding author at: School of Mathematical Sciences, Capital Normal University, Beijing, 100048, China.
E-mail address: gnge@zju.edu.cn (G. Ge).

http://dx.doi.org/10.1016/j.disc.2017.04.021
0012-365X/

In [18], Rees introduced the notation of URDs and showed that all admissible URD$(\{2,3\};v)$ exist with two exceptions. For $K=\{2,4\}$, the URD$(\{2,4\};v)$ has been constructed for most cases in [6,24], with 1338 pairs of (v,r_2) unsettled. For $K=\{3,4\}$, the existence problem has been studied in [5,22–24] and was settled in [27]. We list these results as follows.

Theorem 1.1 *([18]). There exists a URD$(\{2,3\};v)$ with $r_2,r_3>0$ if and only if $v\equiv 0 \pmod 6$ and $r_2+2r_3=v-1$, except for $(v,r_2)=(6,1)$ or $(12,1)$.*

Theorem 1.2 *([27]). There exists a URD$(\{3,4\};v)$ with $r_3,r_4>0$ if and only if $v\equiv 0 \pmod{12}$ and $2r_3+3r_4=v-1$, except for $(v,r_3)=(12,1)$.*

Theorem 1.3 *([24]). There exists a URD$(\{2,4\};v)$ with $r_2,r_4>0$ if and only if $v\equiv 0 \pmod 4$ and $r_2+3r_4=v-1$ except for $(v,r_2)\in\{(8,1),(20,1),(12,2),(12,5)\}$, and possibly excepting:*

- $(v,r_2)=(2n,1)$, $n\in\{52,100,184\}$;
- $(v,r_2)=(2n,r_2)$, $n\in\{34,46,70,82,94,118,130,178,202,214,238,250,334\}$, r_2 admissible;
- $(v,r_2)=(12n,2)$, $n\in\{2,7,9,10,11,13,14,17,19,22,31,34,38,43,46,47,82\}$.

Recently, 4-RGDDs of type 2^u with $u\in\{34,52,184,238\}$ have been constructed in [7,26]. Thus the possible exceptions $(v,r_2)=(2n,1)$ with $n\in\{34,52,184,238\}$ can be removed from Theorem 1.3.

Theorem 1.4 *([3,7,10,11,13,15,16,19,21,24–26]). The necessary conditions for the existence of a k-RGDD of type h^u, namely, $u\geq k$, $hu\equiv 0 \pmod k$ and $h(u-1)\equiv 0 \pmod{k-1}$, are also sufficient for*
$k=2$;
$k=3$, except for $(h,u)\in\{(2,3),(2,6),(6,3)\}$; and for
$k=4$, except for $(h,u)\in\{(2,4),(2,10),(3,4),(6,4)\}$ and possibly excepting:

1. *$h\equiv 2,10 \pmod{12}$: $h=2$ and $u\in\{46,70,82,94,100,118,130,178,202,214,250,334\}$; $h=10$ and $u\in\{4,94\}$; $h=26$ and $u\in\{10,70,82\}$; $h\in\{38,58,74,82,86,94,106\}$ and $u=10$.*
2. *$h\equiv 6 \pmod{12}$: $h=6$ and $u\in\{6,68\}$; $h=18$ and $u\in\{38,62\}$.*
3. *$h\equiv 0 \pmod{12}$: $h=36$ and $u\in\{14,15,18,23\}$.*

In this paper, we continue to study the existence of URD$(\{2,4\};v)$ and obtain the following result.

Theorem 1.5. *There exists a URD$(\{2,4\};v)$ with $r_2,r_4>0$ if and only if $v\equiv 0 \pmod 4$ and $r_2=v-1-3r_4$ except for $(v,r_2)\in\{(8,1),(20,1),(12,2),(12,5)\}$, and possibly excepting:*

- $(v,r_2)=(2n,1)$, $n\in\{46,70,82,94,100,118,130,202,214\}$;
- $(v,r_2)=(12n,2)$, $n\in\{2,9,10,11,13,14,17\}$.

The approach in this paper is somewhat similar to that in [27]. The major difference lies in the source of the master designs for recursive constructions. The master designs in [27] are obtained by manipulating transversal designs, uniform 5-GDDs, and 5-GDDs of type $(4m)^5(4n)^1$, all of which are well studied. However, in this paper as we only got a few input designs to fill in the holes, the group sizes of the master designs are strictly restricted. We have to construct the master designs directly, including non-uniform K-GDDs with $K=\{k\in\mathbb{Z}:k\geq 5, k\equiv 1 \pmod 4\}$ and non-uniform 4-frames.

2. Preliminaries

A group divisible design $(X,\mathcal{G},\mathcal{B})$ is called *frame resolvable* (and is referred to as a *frame*) if its block set $\mathcal{B}$ admits a partition into *holey parallel classes*, each holey parallel class being a partition of $X\setminus H$ for some hole $H\in\mathcal{G}$. The groups in a frame are often referred to as *holes*. The *hole type* of a frame is just its group type as a GDD. It is well known that in a k-frame, each hole must have size a multiple of $k-1$; in fact the number of holey parallel classes with respect to a given hole H is precisely $|H|/(k-1)$.

Theorem 2.1 *([3,4,9,12,16,17,21,26]). The necessary conditions for the existence of a k-frame of type h^u, namely, $u\geq k+1$, $h\equiv 0 \pmod{k-1}$ and $h(u-1)\equiv 0 \pmod k$, are also sufficient for*
$k=2$;
$k=3$; and for
$k=4$, and possibly excepting:

1. *$h=36$ and $u=12$;*
2. *$h\equiv 6 \pmod{12}$ and*
   (a) *$h=6$ and $u\in\{7,23,27,35,39,47\}$;*
   (b) *$h=18$ and $u\in\{15,23,27\}$;*
   (c) *$h\in\{30,66,78,114,150,174,222,246,258,282,318,330,354,534\}$ and $u\in\{7,23,27,39,47\}$;*
   (d) *$h\in\{n:42\leq n\leq 11238\}\setminus\{66,78,114,150,174,222,246,258,282,318,330,354,534\}$ and $u\in\{23,27\}$.*