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# Decomposing 8-regular graphs into paths of length 4

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ABSTRACT

A $T$-decomposition of a graph $G$ is a set of edge-disjoint copies of $T$ in $G$ that cover the edge set of $G$. Graham and Häggkvist (1989) conjectured that any $2\ell$-regular graph $G$ admits a $T$-decomposition if $T$ is a tree with $\ell$ edges. Kouider and Lonc (1999) conjectured that, in the special case where $T$ is the path with $\ell$ edges, $G$ admits a $T$-decomposition $\mathcal{D}$ where every vertex of $G$ is the end-vertex of exactly two paths of $\mathcal{D}$, and proved that this statement holds when $G$ has girth at least $(\ell+3)/2$. In this paper we verify Kouider and Lonc's Conjecture for paths of length 4.



## 1. Introduction

A decomposition of a graph $G$ is a set $\mathcal{D}$ of edge-disjoint subgraphs of $G$ that cover the edge set of $G$. Given a graph $H$, we say that $\mathcal{D}$ is an $H$-decomposition of $G$ if every element of $\mathcal{D}$ is isomorphic to $H$. Ringel [12] conjectured that the complete graph $K_{2\ell+1}$ admits a $T$-decomposition for any tree $T$ with $\ell$ edges. Ringel's Conjecture is commonly confused with the *Graceful Tree Conjecture* that says that any tree $T$ on $n$ vertices admits a labeling $f : V(T) \to \{0, \dots, n-1\}$ such that $\{1, \dots, n-1\} \subseteq \{|f(x) - f(y)| : xy \in E(T)\}$. Since the Graceful Tree Conjecture implies Ringel's Conjecture [13], Ringel's Conjecture holds for many classes of trees such as stars, paths, bistars, caterpillars, and lobsters (see [3,6]). Häggkvist [7] generalized Ringel's Conjecture for regular graphs as follows.

**Conjecture 1.1** (*Graham–Häggkvist, 1989*). *Let $T$ be a tree with $\ell$ edges. If $G$ is a $2\ell$-regular graph, then $G$ admits a $T$-decomposition*

Häggkvist [7] also proved Conjecture 1.1 when $G$ has girth at least the diameter of $T$. For more results on decompositions of regular graphs into trees, see [4,5,8,9]. For the case where $T = P_\ell$ is the path with $\ell$ edges (note that this notation is not standard), Kouider and Lonc [10] improved Häggkvist's result proving that if $G$ is a $2\ell$-regular graph with girth $g \geq (\ell+3)/2$, then $G$ admits a *balanced* $P_\ell$-decomposition $\mathcal{D}$, that is a path decomposition $\mathcal{D}$ where each vertex is the end-vertex of exactly two paths of $\mathcal{D}$. These authors also stated the following strengthening of Conjecture 1.1 for paths.

**Conjecture 1.2** (*Kouider–Lonc, 1999*). *Let $\ell$ be a positive integer. If $G$ is a $2\ell$-regular graph, then $G$ admits a balanced $P_\ell$-decomposition.*

One of the authors [2] proved the following weakening of Conjecture 1.2: for every positive integers $\ell$ and $g$ such that $g \geq 3$, there exists an integer $m_0 = m_0(\ell, g)$ such that, if $G$ is a $2m\ell$-regular graph with $m \geq m_0$, then $G$ admits a $P_\ell$-decomposition $\mathcal{D}$ such that every vertex of $G$ is the end-vertex of exactly $2m$ paths of $\mathcal{D}$. In this paper we prove Conjecture 1.2 in the case $\ell = 4$.

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### 1.1. Notation

A *trail* $T$ is a graph for which there is a sequence $B = x_0 \cdots x_\ell$ of its vertices such that $E(T) = \{x_i x_{i+1} : 0 \le i \le \ell - 1\}$ and $x_i x_{i+1} \ne x_j x_{j+1}$, for every $i \ne j$. Such a sequence $B$ of vertices is called a *tracking* of $T$ and we say that $T$ is the trail *induced* by the tracking $B$. For another example of the use of trackings, we refer to [1]. We say that the vertices $x_0$ and $x_\ell$ are the *final vertices* of $B$; and that $T$ is *closed* if $x_0 = x_\ell$. Given a tracking $B = x_0 \cdots x_\ell$ we denote by $B^-$ the tracking $x_\ell \cdots x_0$. By abuse of notation, we denote by $V(B)$ and $E(B)$ the sets $\{x_0, \ldots, x_\ell\}$ of vertices, and $\{x_i x_{i+1} : 0 \le i \le \ell - 1\}$ of edges of $B$, respectively. Moreover, we denote by $\bar{B}$ the trail $\big(V(B), E(B)\big)$, and by *length* of $B$ we mean the length of $\bar{B}$. We also use $\ell$-*tracking* to denote a tracking of length $\ell$. A set of edge-disjoint trackings $\mathcal{B}$ of a graph $G$ is a *decomposition of* $G$ *into trackings* or, equivalently, a *tracking decomposition* of $G$ if $\cup_{B\in\mathcal{B}} E(B) = E(G)$. If every tracking of $\mathcal{B}$ has length $\ell$, we say that $\mathcal{B}$ is a *decomposition into* $\ell$-*trackings* (an $\ell$-*tracking decomposition*), and if every tracking of $\mathcal{B}$ induces a path, we say that $\mathcal{B}$ is a *decomposition into path-trackings* (a *path tracking decomposition*). For ease of notation, in this work we make no distinction between the trackings $B$ and $B^-$ in the following sense. Suppose $B \in \mathcal{B}$ is a tracking of a trail $T$; when we need to choose a tracking of $T$ we choose between $B$ and $B^-$ conveniently.

We say that a graph $G$ is *Eulerian* if $G$ contains a closed trail that contains all of the edges of $G$. It is clear that a graph $G$ is Eulerian if and only if $G$ is connected and each of its vertices has even degree. We say that a (not necessarily connected) graph $G$ is *even* if every vertex of $G$ has even degree, i.e., a graph is even if and only if each of its components is Eulerian. An *orientation* $O$ of a subset $E'$ of edges of $G$ is an attribution of a direction (from one vertex to the other) to each edge of $E'$. If an edge $xy$ is directed from $x$ to $y$ in $O$, we say that $xy$ *leaves* $x$ and *enters* $y$. Given a vertex $v$ of $G$, we denote by $d^+_O(v)$ (resp. $d^-_O(v)$) the number of edges leaving (resp. entering) $v$ with respect to $O$. In this paper, we are interested in orientations $O$ such that $d^+_O(v) = d^-_O(v)$, for every vertex $v$ of $G$. For ease of notation, we say that such an orientation is an *Eulerian* orientation. Note that we do not require the graph to be connected in order to admit an Eulerian orientation. It is not hard to see that $G$ admits an Eulerian orientation if and only if each of its components is Eulerian, i.e., if $G$ is an even graph. This fact is used frequently in this paper. We also denote by $O^-$, called *reverse orientation*, the orientation of $E'$ such that if $xy \in E'$ is directed from $x$ to $y$ in $O$, then $xy$ is directed from $y$ to $x$ in $O^-$.

Suppose that every tracking in $\mathcal{B}$ has length at least 2. We consider an orientation $O$ of a set of edges of $G$ as follows. For each tracking $B = x_0 \cdots x_\ell$ in $\mathcal{B}$, we orient $x_0x_1$ from $x_1$ to $x_0$, and $x_{\ell-1}x_\ell$ from $x_{\ell-1}$ to $x_\ell$. Given a vertex $v$ of $G$, we denote by $\mathcal{B}(v)$ the number of edges of $G$ directed towards $v$ in $O$ (i.e., $\mathcal{B}(v) = d^-_O(v)$) and by $\mathrm{Hang}(v, \mathcal{B})$ the number of edges leaving $v$ in $O$ (i.e., $\mathrm{Hang}(v, \mathcal{B}) = d^+_O(v)$). We say that an edge that leaves $v$ in $O$ is a *hanging* edge at $v$ (this definition coincides with the definition of *pre-hanging* edge in [1]). We say that a tracking decomposition $\mathcal{B}$ of $G$ is *balanced* if $\mathcal{B}(u) = \mathcal{B}(v)$ for every $u, v \in V(G)$. It is clear that if $\mathcal{B}$ is a balanced path tracking decomposition of $G$, then $\bar{\mathcal{B}}$ is a balanced path decomposition of $G$.

We say that a subgraph $F$ of a graph $G$ is a *factor* of $G$ if $V(F) = V(G)$. If a factor $F$ is $r$-regular, we say that $F$ is an $r$-*factor*. Also, we say that a decomposition $\mathcal{F}$ of $G$ is an $r$-*factorization* if every element of $\mathcal{F}$ is an $r$-factor.

### 1.2. Overview of the proof

Let $G$ be an 8-regular graph. In Section 2 we use Petersen's 2-factorization theorem to obtain a 4-factorization $\{F_1, F_2\}$ of $G$. Then, we prove that $F_1$ admits a balanced $P_2$-decomposition $\mathcal{D}$. Given an Eulerian orientation $O$ to the edges of $F_2$, we *extend* each path $P$ of $\mathcal{D}$ to a trail of length 4 using one outgoing edge of $F_2$ at each end-vertex of $P$ (see Fig. 1), thus obtaining a 4-tracking decomposition $\mathcal{B}$ of $G$. We also prove that these extensions can be chosen such that no element of $\mathcal{B}$ is a cycle of length 4. Lemma 2.7 shows that $O$ can be chosen with some additional properties, which we call *good orientation* (see Definition 2.5), and Lemma 2.8 uses these special properties to show that the elements of $\mathcal{B}$ that do not induce paths can be paired with paths of $\mathcal{B}$ to form a new special element, which we call *exceptional extension* (see Fig. 6). Thus, we can understand $\mathcal{B}$ as a decomposition into paths and exceptional extensions. In Section 3, we show how to switch edges between the elements to obtain a decomposition into paths.

## 2. Decompositions into extensions

In this section we use Petersen's Factorization Theorem [11] to obtain a well-structured tracking decomposition of 8-regular graphs, called *exceptional decomposition into extensions*.

**Theorem 2.1** (*Petersen's 2-Factorization Theorem*)**.** *Every* $2k$*-regular graph admits a* 2*-factorization.*

Let $G$ be an 8-regular graph and let $\mathcal{F}$ be a 2-factorization of $G$ given by Theorem 2.1. By combining the elements of $\mathcal{F}$ we obtain a decomposition of $G$ into two 4-factors, say $F_1$ and $F_2$. From now on, we fix such two 4-factors $F_1$ and $F_2$. In the figures throughout this section (and also in Fig. 10(a)), we illustrate the edges of $F_1$ as dashed edges, and the edges of $F_2$ as straight edges. We first prove the following straightforward lemma.

**Lemma 2.2.** *If* $G$ *is a* 4*-regular graph, then* $G$ *admits a balanced* $P_2$*-decomposition.*