Discrete Mathematics 340 (2017) 2301–2306

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Graphs constructible from cycles and complete graphs

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ARTICLE INFO

Article history:
Received 22 December 2016
Received in revised form 2 May 2017
Accepted 2 May 2017

Keywords:
Cycle
Clique
m-sum
Θ_n-closed
Menger

ABSTRACT

A graph G is Θ_3*-closed* provided any two vertices of G that are joined by three internally disjoint paths are adjacent. We show that the Θ_3-closed graphs are precisely those that can be obtained by gluing together cliques and cycles along vertices or edges.

1. Introduction

Menger's classical theorem on connectivity states that, for any two *nonadjacent* vertices x and y in a graph G, the minimum number of vertices that need to be deleted to disconnect x from y equals the maximum number of internally disjoint x, y-paths. Our focus in this paper comes from a slightly different perspective. We consider graphs with the following property: if there are n internally disjoint x, y-paths in G, then x and y are necessarily adjacent. We call such a graph Θ_n-closed. Our main goal is to characterize such graphs for $n \leq 3$. In Menger's theorem, one has to consider minimum x, y-cuts. For our purposes, we need to consider *all* minimal x, y-cuts, so also the minimal cuts that are not of minimum size. Our main result is that Θ_3-closed graphs can be built from basic building blocks constituted by cycles and cliques. The construction amounts to gluing these building blocks together along vertices or edges, that is, complete subgraphs of order $m \leq 2$. We thought it appropriate to define the concepts and notation for all n and m. Some of the intermediate results could be proved for all values of n and/or m anyway. But it seems that the characterization of Θ_n-closed graphs, for $n \geq 4$, might be quite a challenge.

2. Setting the stage

Let G be a simple graph with vertex set V. Let x and y be two nonadjacent vertices in G. A subset S of V is an x, y*-cut* if the removal of the vertices in S from G results in a graph $G–S$, where x and y are in different components. So all paths between x and y have been broken in $G–S$. The minimum cardinality of such x, y-cuts is denoted by $\kappa(x, y)$. The maximum number of internally disjoint paths between x and y is denoted by $\lambda(x, y)$. Menger's classical theorem states that $\kappa(x, y) = \lambda(x, y)$, see [3,5]. A *minimal* x, y*-cut* is an x, y-cut S such that any proper subset of S does not cut through all x, y-paths. Let $\kappa^*(x, y)$ denote the cardinality of a *largest* minimal x, y-cut. Clearly, we have $\kappa(x, y) \leq \kappa^*(x, y)$. The *connectivity* $\kappa(G)$ of G is the

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http://dx.doi.org/10.1016/j.disc.2017.05.002

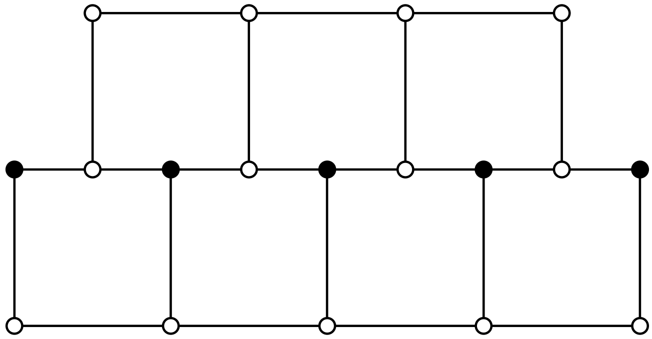

Fig. 1. A graph with $\widetilde{\kappa}(G) < \widetilde{\kappa}^*(G)$.

minimum value of $\kappa(x, y)$ over all nonadjacent pairs of vertices x and y, that is, the minimum cardinality of a vertex cut. This is a classical and well-studied parameter. The connectivity of a graph is a "global" property of the whole graph. On the other hand, the realization of the value of $\kappa(G)$ might be a very local phenomenon. For instance, we might have a vertex z of minimum degree equal to $\kappa(G)$, so that the set of its neighbors is a minimum vertex cut, whereas any nonadjacent pair of vertices x, y distinct from z might need a much larger x, y-cut. Our perspective in this paper is different. We want to combine a local and a global perspective. The values of $\kappa(x, y)$ and $\kappa^*(x, y)$ are local parameters, but we want to have a restriction on these values globally. This leads us to the following new graph parameters.

We define $\widetilde{\kappa}(G)$ to be the *maximum* value of $\kappa(x, y)$ over all nonadjacent pairs of vertices x, y in G. Likewise, $\widetilde{\kappa}^*(G)$ is the *maximum* value of $\kappa^*(x, y)$ over all non-adjacent pairs x and y in G. Clearly, we have $\widetilde{\kappa}(G) \leq \widetilde{\kappa}^*(G)$. This inequality can be strict in general. Indeed, the graph in Fig. 1 has maximum degree 3, so we have $\widetilde{\kappa}(G) \leq 3$, and it is straightforward to verify that we actually have $\widetilde{\kappa}(G) = 3$. On the other hand, the five black vertices form a minimal vertex cut between any bottom vertex and any top vertex, so $\widetilde{\kappa}^*(G) \geq 5$. Again, it is straightforward to verify that we actually have equality here. This construction can be extended in the obvious way to produce graphs G with $\widetilde{\kappa}(G) = 3$ and $\widetilde{\kappa}^*(G)$ arbitrarily large. It is surprising that this cannot be done for $\widetilde{\kappa}(G) = 2$. We will show in Theorem 3 that every graph G with $\widetilde{\kappa}(G) = 2$ also has $\widetilde{\kappa}^*(G) = 2$. This theorem allows us to characterize the graphs G with $\widetilde{\kappa}(G) \leq 2$, which is our main result.

We would like to present our results in a broader context. For this purpose, we introduce the following types of graphs.

- A graph G is a Λ_n*-graph* if $\widetilde{\kappa}(G) \leq n$.
- A graph G is a Λ_n^**-graph* if $\widetilde{\kappa}^*(G) \leq n$.
- A graph G is a Θ_n*-closed graph* if any two vertices that are joined by n internally disjoint paths are necessarily adjacent.

Note that, if G is a Λ_n-graph, then it is also a Λ_m-graph for any $m \geq n$, and likewise for Λ_n^*-graphs and Θ_n-graphs. Moreover, a Λ_n-graph is also a Λ_n^*-graph. The graph in Fig. 1 shows that the converse in general is not true. The reason for the choice of the term *closed* in Θ_n-closed is that we can use this notion to define a *closure system*, sensu [4], on the edges of the complete graph on V. As a mnemonic, the symbol Θ suggests two adjacent vertices joined by two internally disjoint paths. So, it visualizes Θ_2. A graph is Θ_1-closed if and only if it is a disjoint union of complete graphs: if there is a path between x and y, then x and y must be adjacent. Recall that a block in a graph is a maximal connected subgraph that has no cut vertex. Hence, it is either a maximal 2-connected graph or it is an edge, each end vertex of which is a cut vertex or has degree 1. A *block graph* is a connected graph in which each block induces a complete graph [2,3]. A graph is Θ_2-closed if and only if it is the disjoint union of block graphs: just observe that any two vertices on a cycle must be adjacent, so all blocks are complete. In this paper, we give a characterization of the Θ_n-closed graphs, for $n = 3$. Where possible, we prove our results for arbitrary values of n. Before we get to this, we need some more terminology and notation.

As said above, our convention is that V is the vertex set of G. If other graphs are involved, then we denote the vertex set of a graph H by $V(H)$. If a graph H is a subgraph of a graph H', then we denote this by $H \subseteq H'$. The *open neighborhood* $N(v)$ of a vertex v is the set of all neighbors of v, and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs that both contain a complete subgraph with m vertices, say Z_1 in G_1 and Z_2 in G_2. A graph G obtained from G_1 and G_2, by identifying the vertices of Z_1 with the vertices of Z_2, is called an *m-sum* of G_1 and G_2, where G_1 and G_2 are *m-summands* of G. The graphs G_1 and G_2 are, so to speak, *glued together along* a complete subgraph of order m. Note that, in general, we can get non-isomorphic m-sums by this construction. We can describe the m-sum of the two graphs also in a slightly different way that gives us some convenient notation. The *union* of the graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. Likewise, the *intersection* of G_1 and G_2 is the graph $G_1 \cap G_2$ with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$. If $G_1 \cap G_2$ is a complete graph Z with m vertices, then $G_1 \cup G_2$ is an m-sum of G_1 and G_2 over Z, with G_1 and G_2 being the m-summands of the sum. Obviously, the summands of an m-sum G are induced subgraphs of G.

To make some statements more elegant, we will call the graph K_2, by abuse of language, just an *edge*. Then, the trees are precisely the graphs that are obtained from edges by taking 1-sums. The block graphs are precisely the graphs that are obtained from complete graphs by taking 1-sums, and a cactus [1,5] is a graph that can be obtained from edges and cycles by taking 1-sums (every block in a cactus is an edge or a cycle).