ARTICLE IN PRESS

Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Coloring the cliques of line graphs

Gábor Bacsó [a], Zdeněk Ryjáček [b], Zsolt Tuza [c,d]

[a] Computer and Automation Institute, Hungarian Academy of Sciences, H–1111 Budapest, Kende u. 13–17, Hungary
[b] Department of Mathematics, University of West Bohemia, 306 14 Pilsen, Czech Republic
[c] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, H–1053 Budapest, Reáltanoda u. 13–15, Hungary
[d] Department of Computer Science and Systems Technology, University of Pannonia, H–8200 Veszprém, Egyetem u. 10, Hungary

ARTICLE INFO

Article history:
Received 4 April 2016
Accepted 8 November 2016
Available online xxxx

Keywords:
Chromatic number
Weak coloring
Clique chromatic number
Clique chromatic index
Line graph
Ramsey theory

ABSTRACT

The *weak chromatic number*, or *clique chromatic number* (CCHN) of a graph is the minimum number of colors in a vertex coloring, such that every maximal clique gets at least two colors. The *weak chromatic index*, or *clique chromatic index* (CCHI) of a graph is the CCHN of its line graph.

Most of the results here are upper bounds for the CCHI, as functions of some other graph parameters, and contrasting with lower bounds in some cases. Algorithmic aspects are also discussed; the main result within this scope (and in the paper) shows that testing whether the CCHI of a graph equals 2 is NP-complete. We deal with the CCHN of the graph itself as well.

1. Introduction

Covering and coloring (partitioning) are very general fundamental problems in combinatorics. In the present paper we investigate them for inclusionwise maximal complete subgraphs of graphs, mostly in line graphs. Quantitative and also algorithmic results will be presented. For the detailed history of covering and coloring the cliques of graphs, which started in the mid-1980s, we refer to [3] where numerous references are given. Before stating results explicitly, let us introduce the terminology that will be used throughout the paper.

1.1. Definitions and notation

In this paper, by a *graph* we mean a finite graph $G = (V, E)$ without loops or multiple edges; except if we explicitly emphasize the contrary. For a vertex x, $N_G(x)$ denotes the set of neighbors of x in G, and $\Gamma_G(x)$ stands for the induced subgraph on $N_G(x)$. We use $\alpha(G)$ to denote the *independence number* of G, i.e., the maximum size of an independent (stable) set in G, and throughout the paper, 'ln' means the natural logarithm. Throughout the paper, by a *clique* in a graph G we always mean an inclusionwise nontrivial maximal complete subgraph of G.

For a graph G, $L(G)$ is the *line graph of* G, i.e., the vertex set of $L(G)$ is $E(G)$, and two edges e and f are adjacent if and only if they have some common endpoint(s). We emphasize here that in the case of defining line graphs, we allow multiple edges in the original graph G (or, more exactly, we allow G to be a multigraph). Our definition also means that $L(G)$ is defined to be a simple graph (unlike some also frequently used definitions).

A hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is *k-colorable* if the elements of $V(\mathcal{H})$ can be colored with k colors so that no monochromatic hyperedge occurs, except the singletons (if $\mathcal{H}$ contains 1-element edges). The minimum possible k for which $\mathcal{H}$ is k-colorable is called the *chromatic number* of $\mathcal{H}$. In the context of the present work, a *transversal* of a hypergraph is a set

E-mail address: ryjacek@kma.zcu.cz (Z. Ryjáček).

http://dx.doi.org/10.1016/j.disc.2016.11.011

of vertices meeting every *non-singleton hyperedge*; and the minimum size of a transversal in $\mathcal{H}$ is denoted by $\tau(\mathcal{H})$ and called the *transversal number* of $\mathcal{H}$. It is important to emphasize that here we deviate from the standard definition, which would require that a transversal contains all 1-element hyperedges.

In the context of this paper, the following concept is of crucial importance: given a graph G, the *clique hypergraph of* G is the hypergraph $\mathcal{C}(G)$ having $V(G)$ as vertex set, and the (maximal) cliques of G as hyperedges (recall that we have defined cliques in G to be *maximal* complete subgraphs of G). A graph G is *weakly k-colorable* (or *k-clique-colorable*), if its clique hypergraph is k-colorable (or, equivalently, if there is a k-vertex-coloring of G such that there is no monochromatic clique). The *weak chromatic number of* G, denoted $\chi_C(G)$, is the chromatic number of the clique hypergraph of G (or, equivalently, $\chi_C(G)$ is the minimum number k, such that G is weakly k-colorable). Similarly, the *clique-transversal number* $\tau_C(G)$ *of* G is the transversal number of the clique hypergraph of G.

Finally, the *weak chromatic index* $\chi'_C(G)$ *of a graph* G is defined as the weak chromatic number of its line graph, i.e., $\chi'_C(G) := \chi_C(L(G))$. (It is a well-known fact that a clique in $L(G)$ corresponds to a triangle or to a star in G, hence $\chi'_C(G)$ can be equivalently defined as the minimum number k for which there is a k-coloring of edges of G such that there is neither a monochromatic nontrivial star nor a monochromatic triangle.)

In this paper, we show the following:

- in Section 2, we deal with the weak chromatic index $\chi'_C(G)$, and we show
  - NP-completeness of deciding whether $\chi'_C(G) \leq 2$,
  - some upper bounds on $\chi'_C(G)$ in terms of properties of vertex neighborhoods and in terms of the chromatic number $\chi(G)$;
- in Section 3, we consider the weak chromatic number $\chi_C(G)$, and we give an upper bound on χ_C based on the clique transversal number τ_C;
- in Section 4, we consider algorithmic aspects of the above problems, and we show polynomial algorithms giving a weak coloring for some special types of inputs.

1.2. *Former results on sufficient conditions for* $\chi_C(G) = 2$

It is immediate to observe that $\chi_C(G) \leq \chi(G)$ with $\chi_C(G) = \chi(G)$ if G is triangle-free, and that $\chi(G) - \chi_C(G)$ can be arbitrarily large (an example is a complete graph of order $n \geq 2$, for which $\chi(G) = n$ while $\chi_C(G) = 2$). In this paragraph, we mention some cases when a graph is known to be weakly 2-colorable.

Strongly perfect graphs. For these graphs, in every induced subgraph H, there exists a stable set, meeting all the maximal cliques of H, by definition. It can be easily seen that such graphs are perfect. Moreover, they are obviously 2-clique-colorable. Strongly perfect graphs were introduced by Berge and Duchet in 1984 [4]. It is known that the following graphs are strongly perfect:
– perfectly orderable graphs [5,6];
– graphs with Dilworth number at most 3 [12];
– (quasi-) Meyniel graphs [7,10].

Small independence number. From Corollary 2 in [2] (which states that if $G \neq C_5$ and $\alpha(G) \geq 2$, then $\chi_C(G) \leq \alpha(G)$), we immediately have the following result.

Theorem 1. *If* $\alpha(G) = 2$ *and* $G \notin \{C_5, 2K_1\}$, *then* $\chi_C(G) = 2$.

Remark 2. The condition on α cannot be replaced by the local assumption that G is claw-free (neither that G is a line graph) because if $n > ck!$ for a suitably chosen constant c, then $\chi_C(L(K_n)) > k$, from old known results in Ramsey theory. Later on (in Section 2.2.2) we will treat this question in more detail.

Later on, in Theorem 16 and Corollary 17, we will see that $\chi_C(G) = 2$ also if G is the line graph of a 4-colorable graph and not an odd cycle of length at least five.

2. The weak chromatic index

Most of the questions discussed in this paper, are strongly related to the following characterization problem:

WeakLine-k
Given a natural number k, for which graphs H does $\chi'_C(H) \leq k$ hold?

In [1], we can find a method, which reduces the weak edge coloring problem to the coloring problem of the triangle hypergraph of the graph, where triangles are viewed as edge triplets. Thus, we do not have to care of the monochromatic stars, only of the monochromatic triangles. This method is reconsidered in Section 4.1 from algorithmic point of view and a quick algorithm is extracted (Theorem 21 and Algorithm 1).