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Proper connection and size of graphs

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ABSTRACT

An edge-coloured graph G is called *properly connected* if any two vertices are connected by a path whose edges are properly coloured. The *proper connection number* of a connected graph G, denoted by $\mathrm{pc}(G)$, is the smallest number of colours that are needed in order to make G properly connected. Our main result is the following: Let G be a connected graph of order n and $k \geq 2$. If $|E(G)| \geq \binom{n-k-1}{2} + k + 2$, then $\mathrm{pc}(G) \leq k$ except when $k = 2$ and $G \in \{G_1, G_2\}$, where $G_1 = K_1 \vee (2K_1 + K_2)$ and $G_2 = K_1 \vee (K_1 + 2K_2)$.

1. Introduction

We use [12] for terminology and notation not defined here and consider finite and simple graphs only. In particular, if G is a graph, then we denote by $c(G)$ the *circumference* of G, i.e. the order of a longest cycle of G, and by $p(G)$ the *detour number* of G, i.e. the order of a longest path of G.

An edge-coloured graph G is called *rainbow-connected* if any two vertices are connected by a path whose edges have different colours. The concept of rainbow connection in graphs was introduced by Chartrand, Johns, McKeon, and Zhang [3]. The *rainbow connection number* of a connected graph G, denoted by $\mathrm{rc}(G)$, is the smallest number of colours that are needed in order to make G rainbow connected. An easy observation is that if G has n vertices then $\mathrm{rc}(G) \leq n - 1$, since one may colour the edges of a given spanning tree of G with different colours, and colour the remaining edges with one of the already used colours.

As a modification of proper colourings and rainbow connections of graphs, Andrews, Lumduamhom, Laforge, and Zhang [1] and, independently, Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero, and Tuza [2] introduced the concept of proper connections of graphs. An edge-coloured graph G is called *properly connected* if any two vertices are connected by a path whose edges are properly coloured. The *proper connection number* of a connected graph G, denoted by $\mathrm{pc}(G)$, is the smallest number of colours that are needed in order to make G properly connected.

For the proper connection number of graphs, the following results are known.

Proposition 1. *Let G be a connected graph of order n and size m. Then*

(1) $1 \leq \mathrm{pc}(G) \leq \min\{\chi'(G), \mathrm{rc}(G)\}$,
(2) $\mathrm{pc}(G) = 1 \Leftrightarrow$ *G is complete,*

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(3) $\mathrm{pc}(G) = m \Leftrightarrow G \cong K_{1,m}$,
(4) *If* G *is a tree, then* $\mathrm{pc}(G) = \Delta(G)$.
(5) *If* G *is traceable, then* $\mathrm{pc}(G) \le 2$.

For each pair of positive integers n and k, we define by $g(n, k)$ the smallest integer such that every connected graph of order n and size at least $g(n, k)$ has proper connection number at most k. Huang, Li, and Wang [5] showed that $g(n, k) = \binom{n-k-1}{2} + k + 2$ for $k = 2, n \ge 14$, and for $k = 3, n \ge 14$. In this paper we completely determine the function $g(n, k)$.
The analogous problem for rainbow connections was introduced in [7] and results on that problem appear in [6–9,11].

2. Auxiliary results

We shall use the following two results of Borozan et al. [2].

Proposition 2 ([2]). *If a graph* G *contains a vertex* v *such that* $d(v) \ge 2$ *and* $\mathrm{pc}(G - v) \le 2$, *then* $\mathrm{pc}(G) \le 2$.

Theorem 1 ([2]). *If* G *is a* 2*-connected graph, then* $\mathrm{pc}(G) \le 3$.

Huang et al. [5] extended Theorem 1 as follows.

Theorem 2 ([5]). *If* G *is a connected bridgeless graph, then* $\mathrm{pc}(G) \le 3$.

For a connected graph G with bridges, Huang et al. [5] described the following natural approach: Let $B \subseteq E(G)$ be the set of bridges of G and let G^* be the graph obtained from G by contracting each component of $G - B$ to a single vertex. Then G^* is obviously a tree with edge set B. They then proved the following result.

Theorem 3 ([5]). *If* G *is a connected graph, then* $\mathrm{pc}(G) \le \max\{3, \Delta(G^*)\}$.

We shall repeatedly use the following identities.

Proposition 3. *For every pair of positive integers* a *and* b,

$$\binom{a}{2} + \binom{b}{2} = \binom{a+b}{2} - ab, \quad \text{and} \tag{1}$$

$$\binom{a+1}{2} = \binom{a}{2} + a. \tag{2}$$

The next lemma will be useful for the proof of our main result.

Lemma 1. *Let* G *be a connected graph with* t *bridges. Then*

$$|E(G)| \le \binom{n-t}{2} + t.$$

Proof. The proof is by induction on t. The result obviously holds when $t = 0$. Now assume $t \ge 1$. Then $G - B$ has $t + 1$ components. Let H be a component of $G - B$ that corresponds to a leaf of G^*. Let $n(H) = r$. Then $G - V(H)$ is a connected graph of order $n - r$ with $t - 1$ bridges. Hence, by our induction hypothesis, $E(G - V(H)) \le \binom{n-r-(t-1)}{2} + t - 1$. Therefore, using Proposition 3, we obtain

$$\begin{aligned} |E(G)| &\le \binom{n-r-t+1}{2} + t - 1 + \binom{r}{2} + 1 \\ &= \binom{n-t+1}{2} - r(n-r-t+1) + t \\ &= \binom{n-t}{2} + (n-t) - r(n-r-t) - r + t \\ &= \binom{n-t}{2} + (n-r-t)(1-r) + t \\ &\le \binom{n-t}{2} + t \end{aligned}$$

since $n \ge t + r$ and $r \ge 1$. □

We end this section with some results on the existence of long cycles in graphs that we need for the proof of our main theorem. We first state a classic result of Erdős and Gallai [4].

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