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On interval and cyclic interval edge colorings of (3, 5)-biregular graphs

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Dedicated to the memory of Horst Sachs.

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ABSTRACT

A proper edge coloring f of a graph G with colors $1, 2, 3, \dots, t$ is called an interval coloring if the colors on the edges incident to every vertex of G form an interval of integers. The coloring f is cyclic interval if for every vertex v of G, the colors on the edges incident to v either form an interval or the set $\{1, \dots, t\} \setminus \{f(e) : e \text{ is incident to v}\}$ is an interval. A bipartite graph G is (a, b)-biregular if every vertex in one part has degree a and every vertex in the other part has degree b; it has been conjectured that all such graphs have interval colorings. We prove that every $(3, 5)$-biregular graph has a cyclic interval coloring and we give several sufficient conditions for a $(3, 5)$-biregular graph to admit an interval coloring.

1. Introduction

An *interval coloring* (or *consecutive coloring*) of a graph G is a proper coloring of the edges by positive integers such that the colors on the edges incident to any vertex of G form an interval of integers. The notion of interval colorings was introduced by Asratian and Kamalian [5] (available in English as [6]), motivated by the problem of finding compact school timetables, that is, timetables such that the lectures of each teacher and each class are scheduled at consecutive periods. Hansen [14] described another scenario (originally suggested by Jesper Bang-Jensen). A school wishes to schedule parent–teacher conferences in time slots so that every person's conferences occur in consecutive slots. A solution exists if and only if the bipartite graph with vertices for the people and edges for the required meetings has an interval coloring.

All regular bipartite graphs have interval colorings, since they decompose into perfect matchings. Not every graph has an interval coloring, since a graph G with an interval coloring must have a proper $\Delta(G)$-edge-coloring [5], where $\Delta(G)$ denotes the maximum degree of a graph G. Sevastjanov [26] proved that determining whether a bipartite graph has an interval coloring is $\mathcal{NP}$-complete. Nevertheless, trees [6,14,18], complete bipartite graphs [5,14,18], grids [12], and outerplanar bipartite graphs [7,13] all have interval colorings. Giaro [11] showed that one can decide in polynomial time whether bipartite graphs with maximum degree 4 have interval 4-colorings. The smallest known maximum degree of a bipartite graph without an interval coloring is 11 [23].

A bipartite graph with parts X, Y is called (a, b)*-biregular* if all vertices of X have degree a and all vertices of Y have degree b. In this paper we study the following well-known conjecture [17,27] for the case $(a, b) = (3, 5)$:

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Conjecture 1.1. *Every* (a, b)*-biregular graph has an interval coloring.*

By results of [14] and [16], all $(2, b)$-biregular graphs admit interval colorings (the result for odd b was obtained independently by Kostochka [19]). Hanson and Loten [15] proved that no (a, b)-biregular graph has an interval coloring with fewer than $a + b - \gcd(a, b)$ colors, where gcd denotes the greatest common divisor.

Several sufficient conditions for a $(3, 4)$-biregular graph G to admit an interval 6-coloring have been obtained: Pyatkin [25] proved that if G has a 3-regular subgraph covering the vertices of degree 4, then it has an interval coloring; Yang et al. [28] proved that if G is the union of two edge-disjoint $(2, 3)$-biregular subgraphs H_1 and H_2 such that vertices of degree 4 in G have degree 2 in H_1 and H_2, then G has an interval coloring; G has an interval coloring if it has a spanning subgraph consisting of paths with endpoints at 3-valent vertices and lengths in $\{2, 4, 6, 8\}$ [3,8]. (See also [2,9] for related results.) However, it is still an open question whether every $(3, 4)$-biregular graph has an interval coloring. In [10] the first and third authors proved that every $(3, 6)$-biregular graph has an interval 7-coloring; by the result in [1] the number of colors is best possible.

It is unknown whether all $(3, 5)$-biregular graphs have interval colorings; to the best of our knowledge no non-trivial condition implying interval colorings of such graphs is known. By the result of Hanson and Loten [15] we need at least 7 colors for an interval coloring of a $(3, 5)$-biregular graph. In this paper we present a technique for constructing interval 7-colorings of families of $(3, 5)$-biregular graphs using so-called *MP-subgraphs* (defined in Section 3). Using this technique we give several sufficient conditions for a $(3, 5)$-biregular graph to admit an interval 7-coloring. Moreover, we present infinite families of $(3, 5)$-biregular graphs satisfying our conditions.

First, we consider cyclic interval colorings. A proper edge coloring f of a graph G with colors $1, 2, 3, \dots, t$ is *cyclic interval* if for every vertex v of G $\{f(e) : e \text{ is incident to } v\}$ or $\{1, \dots, t\} \setminus \{f(e) : e \text{ is incident to } v\}$ is an interval. Cyclic interval colorings are studied in e.g. [20,21,24]. In particular, the general question of determining whether a bipartite graph G has a cyclic interval coloring is $\mathcal{NP}$-complete [20] and there are concrete examples of connected bipartite graphs having no cyclic interval coloring [21]. Trivially, any bipartite graph with an interval coloring also has a cyclic interval coloring with $\Delta(G)$ colors, but the converse does not hold [21]. This means, using the result of Hanson and Loten [15], that Conjecture 1.1 has the following weaker consequence for which the answer is unknown.

Conjecture 1.2. *For any t satisfying that* $\max\{a, b\} \le t \le a + b - \gcd(a, b)$*, every* (a, b)*-biregular graph has a cyclic interval* t*-coloring.*

Note that all bipartite graphs G with $\Delta(G) - \delta(G) \le 1$ admit cyclic interval colorings [10], where $\delta(G)$ as usual denotes the minimum degree in G; so the smallest unsolved case of Conjecture 1.2 is $(a, b) = (3, 5)$. In [10] the first and third authors proved that all $(4, 8)$-biregular graphs have cyclic interval 8-colorings. In this paper we prove that all $(3, 5)$-biregular graphs have cyclic interval 6-colorings, and we give several sufficient conditions for a $(3, 5)$-biregular graph to admit such a coloring with 5 colors.

The rest of the paper is organized as follows. In Section 2 we consider cyclic interval colorings, Section 3 contains sufficient conditions for interval coloring $(3, 5)$-biregular graphs and in Section 4 we discuss these conditions and provide infinite families of graphs satisfying our conditions.

2. Cyclic interval edge colorings

In this section we prove our results on cyclic interval colorings. First we introduce some notation and also state some preliminary results. Throughout the paper, we use the notation $G = (X, Y; E)$ for a bipartite graph G with bipartition (X, Y) and edge set $E = E(G)$. We use the convention that if $G = (X, Y; E)$ is (a, b)-biregular, then the vertices in X have degree a. We denote by $d_G(v)$ the degree of a vertex v in G, and by $N_G(v)$ the set of vertices adjacent to v in G. If $V' \subseteq V(G)$, then $N_G(V') = \bigcup_{v \in V'} N_G(v)$.

For an edge coloring φ of a graph G, let $M(\varphi, i) = \{e \in E(G) : \varphi(e) = i\}$. We define $G_\varphi(a, b) = G[M(\varphi, a) \cup M(\varphi, b)]$. If $e \in M(\varphi, i)$, then e is *colored i under* φ. If φ is a proper t-edge coloring of G and $1 \le a, b \le t$, then a path (cycle) in $G_\varphi(a, b)$ is called a $\varphi - (a, b)$*-colored path (cycle)* in G. We also say that such a path or cycle is φ*-bicolored*. By switching colors a and b on a connected component of $G_\varphi(a, b)$, we obtain another proper t-edge coloring of G. We call this operation a φ*-interchange*. For a vertex $v \in V(G)$, we say that a color i *appears at* v *under* φ if there is an edge e incident to v with $\varphi(e) = i$, and we set

$$\varphi(v) = \{\varphi(e) : e \in E(G) \text{ and } e \text{ is incident to } v\}.$$

If $c \notin \varphi(v)$, then c is *missing at* v *under* φ. Moreover, if $\varphi(v) = \{c\}$, that is, $\varphi(v)$ is singleton, then $\varphi(v)$ usually denotes the color c rather than the set $\{c\}$. In all the above definitions, we often leave out the explicit reference to a coloring φ, if the coloring is clear from the context.

We shall say that a proper t-edge coloring φ of a graph G using positive integers as colors is *interval at a vertex* $v \in V(G)$ if the colors on the edges incident to v form a interval of integers. The coloring φ is *cyclic interval at a vertex* $v \in V(G)$, if it is interval at v or if the set $\{1, \dots, t\} \setminus \{\varphi(e) : e \text{ is incident to } v\}$ is an interval of integers.

A *mixed graph* is a graph containing both directed and undirected edges. We denote a mixed graph G by $G = (V; E, A)$, where $E = E(G)$ are the (undirected) edges of G and $A = A(G)$ are the directed edges or arcs of G. For a mixed graph $G = (V; E, A)$, and a subset $E' \subseteq E$ an *orientation* of E' is the mixed graph obtained from G by orienting each edge of E'.

Let us now prove the main result of this section.