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# The minimum vertex degree for an almost-spanning tight cycle in a 3-uniform hypergraph

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ABSTRACT

We prove that any 3-uniform hypergraph whose minimum vertex degree is at least $\left(\frac{5}{9}+o(1)\right)\binom{n}{2}$ admits an almost-spanning tight cycle, that is, a tight cycle leaving $o(n)$ vertices uncovered. The bound on the vertex degree is asymptotically best possible. Our proof uses the hypergraph regularity method, and in particular a recent version of the hypergraph regularity lemma proved by Allen, Böttcher, Cooley and Mycroft.


## 1. Introduction

The study of Hamilton cycles in graphs and hypergraphs has been an active area of research for many years, going back to Dirac's celebrated theorem [5] that any graph on $n \geq 3$ vertices with minimum degree at least $n/2$ admits a Hamilton cycle. Finding analogues of this theorem for $k$-graphs (*i.e.* $k$-uniform hypergraphs) is one of the major directions of recent research in this area. To discuss these results we use several commonly-used standard terms, definitions of which can be found in Section 2. For a more expository presentation of recent research in this area we refer the reader to the surveys of Kühn and Osthus [17], Rödl and Ruciński [19] and Zhao [27].

The first analogues of Dirac's theorem for $k$-graphs were expressed in terms of minimum codegree, beginning with the work of Katona and Kierstead [13] who established the first non-trivial bounds on the codegree Dirac threshold for a tight Hamilton cycle in a $k$-graph for $k \geq 3$. Rödl, Ruciński and Szemerédi [22,23] then improved this bound by determining the asymptotic value of this threshold, first for $k = 3$ and then for any $k \geq 3$. The asymptotic codegree Dirac threshold for an $\ell$-cycle for any $1 \leq \ell < k$ such that $\ell$ divides $k$ follows as a consequence of this. This left those values of $\ell$ for which $k - \ell$ does not divide $k$, in which cases the Dirac threshold was determined asymptotically through a series of works by Kühn and Osthus [16], Keevash, Kühn, Mycroft and Osthus [14], Hàn and Schacht [9] and Kühn, Mycroft and Osthus [15]. These results can all be collectively described by the following theorem, whose statement gives the asymptotic codegree Dirac threshold for any $k$ and $\ell$.

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**Theorem 1.1** *([9,14–16,22,23]). For any $k \geq 3$, $1 \leq \ell < k$ and $\eta > 0$, there exists $n_0$ such that if $n \geq n_0$ is divisible by $k - \ell$ and $H$ is a $k$-graph on $n$ vertices with*

$$\delta(H) \geq \begin{cases} \left(\frac{1}{2} + \eta\right) n & \text{if } k - \ell \text{ divides } k, \\ \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k - \ell)} + \eta\right) n & \text{otherwise,} \end{cases}$$

*then $H$ contains a Hamilton $\ell$-cycle. Moreover, in each case this condition is best possible up to the $\eta n$ error term.*

More recently the exact codegree Dirac threshold (for large $n$) has been identified in some cases, namely for $k = 3$, $\ell = 2$ by Rödl, Ruciński and Szemerédi [24], for $k = 3$, $\ell = 1$ by Czygrinow and Molla [4], for any $k \geq 3$ and $\ell < k/2$ by Han and Zhao [11], and for $k = 4$ and $\ell = 2$ by Garbe and Mycroft [7].

For other types of degree conditions much less is known. Buß, Hàn and Schacht [3] identified asymptotically the vertex degree Dirac threshold for a loose Hamilton cycle in a 3-graph, following which Han and Zhao [12] established this threshold exactly (for large $n$). Very recently the asymptotic $(k - 2)$-degree Dirac threshold for a Hamilton $\ell$-cycle in a $k$-graph for $1 \leq \ell < k/2$ was determined by Bastos, Mota, Schacht, Schnitzer and Schulenburg [2].

Perhaps the most prominent outstanding open problem is to identify the vertex degree Dirac threshold for a tight Hamilton cycle in a 3-graph. There is a qualitative difference between this problem and the results described above, in that vertex degree conditions only give information about edges containing a given vertex (in particular, many pairs may have degree zero), but a tight cycle requires us to find edges intersecting in pairs. Possibly due to this, progress on this problem has been slower. Glebov, Person and Weps [8] gave the first non-trivial upper bound on this threshold, showing it to be at most $(1 - \varepsilon)\binom{n}{2}$, where $\varepsilon \approx 5 \times 10^{-7}$. Rödl and Ruciński [20] subsequently improved this bound to around $0.92\binom{n}{2}$, and very recently Rödl, Ruciński, Schacht and Szemerédi [21] strengthened this result by showing that the threshold is at most $0.8\binom{n}{2}$. However, this is still some way from the conjectured value of $(\frac{5}{9} + o(n))\binom{n}{2}$ (which matches the lower bound provided by the best known construction).

**Conjecture 1.2** *([19], Conjecture 2.18). For every $\eta > 0$, there exists $n_0$ such that if $H$ is a 3-graph on $n \geq n_0$ vertices with $\delta(H) \geq \left(\frac{5}{9} + \eta\right)\binom{n}{2}$, then $H$ contains a tight Hamilton cycle.*

Actually, the original conjecture was that for any $k \geq 3$ the vertex degree threshold for a tight Hamilton cycle in a $k$-graph is asymptotically equal to that which forces a perfect matching. However, this was disproved for $k \geq 4$ by Han and Zhao [10]. The main result of this paper is the following theorem, which states that the assumptions of Conjecture 1.2 suffice to ensure the existence of an almost-spanning tight cycle.

**Theorem 1.3.** *For every $\eta > 0$ there exists $n_0$ such that if $H$ is a 3-graph on $n \geq n_0$ vertices with $\delta(H) \geq \left(\frac{5}{9} + \eta\right)\binom{n}{2}$ then $H$ contains a tight cycle of length at least $(1 - \eta)n$.*

The following example shows that Conjecture 1.2 and Theorem 1.3 are asymptotically best possible. Fix $\eta > 0$, take disjoint sets $A$ and $B$ with $|A| = \lfloor\frac{(1-\eta)n-1}{3}\rfloor$ and $|B| = n - |A|$, and take $H$ to be the 3-graph on vertex set $A \cup B$ whose edges are all triples which intersect $A$. Then for any tight cycle $C$ in $H$, any three consecutive vertices in $C$ must include a vertex of $A$, so the length of $C$ is at most $3|A| < (1 - \eta)n$. However, it is easily checked that $\delta(H) = \binom{n-1}{2} - \binom{|B|-1}{2} \approx (\frac{5}{9} - \frac{4}{9}\eta - \frac{\eta^2}{9})\binom{n}{2}$.

Many of the Dirac thresholds mentioned above were established by absorbing arguments consisting of two parts: a long cycle lemma which states that the given $k$-graph contains an almost-spanning cycle, and an absorbing lemma which allows us to include in this cycle a so-called absorbing path, which can 'absorb' the small set of leftover vertices to transform the almost-spanning cycle into a Hamilton cycle. In this way Theorem 1.3 represents a significant step towards a full proof of Conjecture 1.2. Furthermore, the proof of Theorem 1.3 provides a clear illustration of how the recent 'Regular Slice Lemma' of Allen, Böttcher, Cooley and Mycroft [1] may be used to prove embedding results for uniform hypergraphs. In our opinion the proof of Theorem 1.3 is significantly more concise and notationally simpler than it would have been using previous hypergraph regularity methods.

While we were finalising this paper, Reiher, Rödl, Ruciński, Schacht and Szemerédi [18] announced a proof of Conjecture 1.2, *i.e.* that the minimum degree condition we consider in fact guarantees a tight Hamilton cycle. Their proof uses the absorbing method, and is very different from the approach used in this paper.

This paper is organised as follows. In Section 2 we give the necessary definitions, then in Section 3 we prove that any 3-graph satisfying the minimum vertex degree condition of Theorem 1.3 admits a tightly-connected perfect fractional matching (Lemma 3.3). In Section 4 we introduce the hypergraph regularity theory we use from [1], including the Regular Slice Lemma, which returns a 'reduced $k$-graph' which almost inherits the minimum vertex degree of the original $k$-graph, and the Cycle Embedding Lemma, which given a large tightly-connected fractional matching in the reduced $k$-graph, returns