Discrete Mathematics 340 (2017) 1180–1186

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the topology of infinite regular and chiral maps

John A. Arredondo [a], Camilo Ramírez Maluendas [a], Ferrán Valdez [b,*]

[a] *Fundación Universitaria Konrad Lorenz, C.P. 110231, Bogotá, Colombia*
[b] *Centro de Ciencias Matemáticas, UNAM, Campus Morelia, C.P. 58190, Morelia, Michoacán, Mexico*

ARTICLE INFO

Article history:
Received 26 July 2015
Received in revised form 5 December 2016
Accepted 30 December 2016

Keywords:
Infinite surface
Regular and chiral polytope

ABSTRACT

We prove that infinite regular and chiral maps can only exist on surfaces with one end. Moreover, we prove that an infinite regular or chiral map on an orientable surface with positive genus, can only be realized on the Loch Ness monster, that is, the topological surface of infinite genus with one end.

1. Introduction

This paper is motivated by the following problem, posed by D. Pellicer in [12]: *determine which non-compact surfaces without boundary admit embeddings of chiral maps*. The present work gives a complete answer to this question and generalizes previous results on minimal regular covers of the Archimedean tessellations, see [1]. More precisely:

Theorem 1.1. *Let $\mathcal{M}$ be a regular or chiral map on a surface S, and let $Aut(\mathcal{M})$ be the automorphism group of the map $\mathcal{M}$. Then the spaces $Ends(S)$ and $Ends(Aut(\mathcal{M}))$ are homeomorphic. In particular, $Aut(\mathcal{M})$ is infinite if and only if $Ends(Aut(\mathcal{M}))$ has one element.*

Here $Ends(S)$ denotes the space of ends of the surface S and $Ends(Aut(\mathcal{M}))$ the space of ends of the Cayley graph of $Aut(\mathcal{M})$. For a precise definition of these spaces see Section 2. The preceding theorem tells us that there is a considerable topological restriction for a surface to support an infinite chiral or regular map, namely, it has to have one end. We also describe the topology of orientable surfaces supporting orientable and chiral maps:

Theorem 1.2. *Let $\mathcal{M}$ be a regular map on a non-compact and orientable surface S. Then S is homeomorphic to either the plane or the Loch Ness monster.*

Theorem 1.3. *Let $\mathcal{M}$ be a chiral map on a non-compact surface S. Then S is homeomorphic to the Loch Ness monster.*

The Loch Ness monster is the only orientable topological surface with infinite genus and only one end. As a mathematical object, this surface appears naturally in many contexts, see *e.g.*, [2,4,20] and [21]. It is important to remark that if an infinite Cayley graph is not planar, then it cannot be embedded in any surface of finite genus, see [7]. On the other hand, it is possible to construct vertex-transitive graphs on the Cantor sphere using a very nice construction (tree amalgamation of graphs) due to B. Mohar (see [9]). Our results imply that these embedded graphs do not define regular or chiral maps.

* Corresponding author.
E-mail addresses: alexander.arredondo@konradlorenz.edu.co (J.A. Arredondo), camilomaluendas@gmail.com (C.R. Maluendas), ferran@matmor.unam.mx (F. Valdez).

http://dx.doi.org/10.1016/j.disc.2016.12.023
0012-365X/

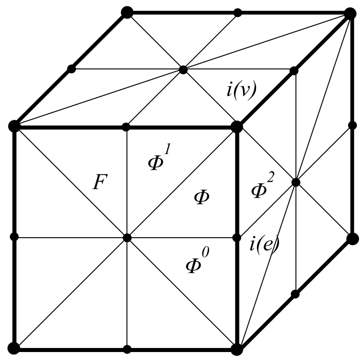

Fig. 1. A map in a cube divided into flags.

2. Preliminaries

Maps. We begin this section by discussing some general aspects of maps that will be needed for the proofs of the main theorems. Our text is not self-contained, hence we refer the reader to [6,12] and references within for details.

In this text the term *surface* means connected 2-dimensional topological real manifold *with empty boundary*, and will be denoted by S. In particular, the transition functions of the corresponding atlas are only required to be continuous. It is important to remark that we do not require S to be a compact topological space. A **map** $\mathcal{M}$ on a surface S is a *finite* 2-cell embedding $i : \Gamma \hookrightarrow S$ of a **locally finite simple** graph Γ into S. In other words, only finitely many edges are incident on each vertex of Γ, the endpoints of each edge are always in different vertices and the function i is a topological embedding such that *each* connected component of $S \setminus i(\Gamma)$ is homeomorphic to a disk bounded by a closed (finite) path in Γ. We denote such a triple by $\mathcal{M} := \mathcal{M}(\Gamma, i, S)$.

Every $f \in Homeo(S)$ for which there exists an automorphism $\rho_f : \Gamma \to \Gamma$ of the graph Γ such that $i \circ \rho_f = f \circ i$ is called a **preautomorphism** of the map $\mathcal{M}(\Gamma, i, S)$. The set of preautomorphisms of a map $\mathcal{M}$ has a natural group structure and we denote it by $\widetilde{Aut}(\mathcal{M})$. It is not difficult to see that for each $f \in \widetilde{Aut}(\mathcal{M})$ the automorphism ρ_f is unique, hence we have a well-defined group morphism $\varphi : \widetilde{Aut}(\mathcal{M}) \to Aut(\Gamma)$, where the codomain is the group of automorphisms of Γ. An **automorphism of the map** $\mathcal{M}$ is an element of the group $Aut(\mathcal{M}) := \widetilde{Aut}(\mathcal{M})/Ker(\varphi)$. By definition, all homeomorphisms of S in a coset $[f] \in Aut(\mathcal{M})$ determine the same graph automorphism ρ_f. We shall abuse notation and we shall write the coset $[f] \in Aut(\mathcal{M})$ as f.

A **flag** Φ of a map $\mathcal{M}(\Gamma, i, S)$ is a triangle on S whose vertices are **vertex** $i(v)$, $v \in V(\Gamma)$, the midpoint of an **edge** $i(e)$ incident to $i(v)$ with $e \in E(\Gamma)$, and an interior point of a **face** $F \subset S \setminus i(\Gamma)$ whose boundary contains $i(e)$. We assume that all flags contained in the closure of a face F share the same interior point of F as vertex. In this way, every map induces a triangulation of S given by its flags. Combinatorially, every flag can be identified with an ordered incident triple formed by a vertex, an edge and a face of the map $\mathcal{M}$. Henceforth, we shall abuse notation and we shall understand flags either as triangles or as ordered triples. Given a flag Φ of the map $\mathcal{M}$, there is a unique adjacent flag Φ^0 (resp. Φ^1 and Φ^2) of the map $\mathcal{M}$ that differs from Φ precisely on the vertex (resp. on the edge and on the face). The flag Φ^j is called the j-**adjacent flag of** Φ. In Fig. 1, we show an example of the cube with some flags marked with their name. We denote the set of flags of a given map $\mathcal{M}$ by $\mathcal{F} := \mathcal{F}(\Gamma, i, S)$. The group $Aut(\mathcal{M})$ acts on the set of flags $\mathcal{F}(\Gamma, i, S)$ and this action is free: every element of $Aut(\mathcal{M})$ is completely determined by the image of a given flag (see Lemma 3.1 in [5]).

Definition 2.1 (*Regular and Chiral Maps*)**.** A map $\mathcal{M}$ is called **regular**, respectively **chiral**, if the action of $Aut(M)$ on $\mathcal{F}$ induces one orbit in flags, respectively the action of $Aut(M)$ on $\mathcal{F}$ induces two orbits in flags with the property that adjacent flags belong to different orbits.

The graph Γ of a regular or chiral map $\mathcal{M}(\Gamma, i, S)$ is always regular, that is, every vertex has the same degree $q \in \mathbb{N}$. Moreover, such maps also satisfy that the boundary of each face in $S \setminus i(\Gamma)$ is formed by a closed path in $i(\Gamma)$ of fixed length p. The pair $\{p, q\}$ is called the **Schläfli type** of the map $\mathcal{M}$. When $\mathcal{M}$ is a regular map the group $Aut(\mathcal{M})$ is generated by three involutions ρ_0, ρ_1, and ρ_2, where ρ_j is the unique automorphism of $\mathcal{M}$ sending a fixed **base flag** Φ to its j-adjacent flag Φ^j. Moreover the generating set $\{\rho_0, \rho_1, \rho_2\}$ satisfies the relations:

$$\rho_1^2 = \rho_1^2 = \rho_1^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = Id, \tag{1}$$

and probably some more (see [12]). On the other hand, when $\mathcal{M}$ is a chiral map the group $Aut(\mathcal{M})$ is generated by two elements $\{\sigma_1, \sigma_2\}$ that satisfy the relations:

$$\sigma_1^p = \sigma_2^q = (\sigma_1\sigma_2)^2 = Id, \tag{2}$$