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Disjoint cycles in graphs with distance degree sum conditions[☆]

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ABSTRACT

In this paper, we prove that for any positive integer k and for any simple graph G of order at least $3k$, if $d(x)+d(y) \geq 4k$ for every pair of vertices x and y of distance 2 in G, then with one exception, G contains k disjoint cycles. This generalizes a former result of Corrádi and Hajnal (1963).

1. Introduction

Let G be a graph. A set of subgraphs of G is said to be disjoint if no two of them have any common vertex in G. Corrádi and Hajnal [2] investigated the maximum number of disjoint cycles in a graph. They proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k disjoint cycles. In particular, when the order of G is exactly $3k$, then G contains k disjoint triangles. One may find a number of results on disjoint cycles in [1–8]. In particular, Enomoto [4] and Wang [8] improved this result by proving the following theorem.

Theorem 1.1 (*[4] and [8]*)**.** *Let G be a graph of order at least $3k$, where k is a positive integer. Suppose that $d(x)+d(y) \geq 4k-1$ for every pair of non-adjacent vertices x and y of G. Then G contains k disjoint cycles.*

In this paper, we will improve the result of Corrádi and Hajnal [2] in a different direction. To state our result, we define a set $F_{l,k,n}$ of graphs in the following.

Let l, k and n be three positive integers such that l is odd and $n-sk+1$ is even, where $s=(l+1)/2$. We define a set $F_{l,k,n}$ of graphs as follows. A graph G belongs to $F_{l,k,n}$ if and only if $V(G)$ has a partition (X, Y, Z) with $|X|=sk-1$ and $|Y|=|Z|=(n-sk+1)/2$ such that the edges between Y and Z consist of $(n-sk+1)/2$ independent edges, there are no edges between X and Z and every vertex in X is adjacent to every vertex in Y. Furthermore, there is no restriction to edges among vertices in X. It is easy to see that $d(u)+d(v) \geq min\{2sk, (n-sk+1)/2+1\}$ for all $\{u, v\} \subseteq V(G)$ with $dist(u, v)=2$ and if $\{u, v\} \subseteq Y$ then $d(u)+d(v)=2sk$. Moreover, each cycle of order at least l in G contains at least s vertices of X and so G does not have k disjoint cycles of order at least l.

We propose the following conjecture:

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Conjecture 1.2. *Let l, k and n be three positive integers such that l is odd and $n \geq lk$. Let G be a connected graph of order n. Suppose that $d(x) + d(y) \geq (l+1)k$ for every pair of vertices x and y of distance 2 in G. Then G contains k disjoint cycles of order at least l or $n - (l+1)k/2 + 1$ is even and G belongs to $F_{l,k,n}$.*

In this paper, we prove the following result to support this conjecture:

Theorem 1.3. *Let G be a connected graph of order at least $3k$, where k is a positive integer. Suppose that $d(x) + d(y) \geq 4k$ for every pair of vertices x and y of distance 2 in G. Then G contains k disjoint cycles or n is odd and G belongs to $F_{3,k,n}$.*

This result improves the aforementioned theorem of Corrádi and Hajnal [2] within connected graphs. A special case of Theorem 1.3 for disjoint triangles is as follows:

Corollary 1.4. *Let G be a connected graph of order $n = 3k$, where k is a positive integer. Suppose that $d(x)+d(y) \geq 4k$ for every pair of vertices x and y of distance 2 in G. Then G contains k disjoint triangles.*

We discuss only finite simple graphs and use standard terminology and notation from [1] except otherwise indicated. For a vertex $u \in V(G)$ and a subgraph H of G, $N(u, H)$ is a set of neighbors of u contained in H. We let $d(u, H) = |N(u, H)|$. Thus $d(u, G)$, simply denoted by $d(u)$, is the degree of u in G. Given two disjoint subgraphs G_1 and G_2 of G, $\sum_{x \in V(G_1)} d(x, G_2)$, written as $e(G_1, G_2)$ sometimes for simplicity, is the number of edges of G between G_1 and G_2. Let U be a subset of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. For a vertex $v \in V(G)$, we write $U + v$ and $U - v$ for $U \cup \{v\}$ and $U \backslash \{v\}$, respectively. We shall use $G \supseteq mK_3$ to represent that G contains a set of m disjoint triangles. The length of a cycle C is denoted by $l(C)$. The distance between two vertices x and y is the length of a shortest xy-path and denoted by $dist(x, y)$.

Let H be a subgraph of G and let $u \in V(H)$ and $x \in G - V(H)$. We write $x \to (H, u)$ if $H - u + x$ has a cycle and otherwise we write $x \nrightarrow (H, u)$.

We organize the paper as follows. In Section 2, we list some useful lemmas. In Section 3, we prove our main theorem.

2. Lemmas

Throughout this section, G denotes a graph of order $n \geq 3$. The lemmas below have already been proved in [8].

Lemma 2.1 ([8]). *Let C be a cycle of length at least 4 and x a vertex of G not on C. If $d(x, C) \geq 2$, then $G[V(C) + x]$ contains a cycle of length less than $l(C)$ unless $d(x, C) = 2$, $l(C) = 4$ and x is adjacent to two non-adjacent vertices of C.*

Lemma 2.2 ([8]). *Let T be a triangle of G. Let x and y be two vertices of G not on T. Suppose that $d(x, T) + d(y, T) \geq 5$. Then T has a vertex z such that $T - z + x$ is a triangle and $yz \in E$.*

Lemma 2.3 ([8]). *Let $P = uvw$ be a path and T a triangle of G such that T is disjoint from P. Then the following hold:*

(a) *If $d(u, T) + d(w, T) \geq 5$ and $d(v, T) \geq 1$, then $G[V(P \cup T)]$ contains two disjoint triangles.*

(b) *If $d(u, T) + d(v, T) + d(w, T) \geq 7$, then either $G[V(P \cup T)]$ contains two disjoint triangles, or $d(u, T) = d(w, T) = 2$, $N(u, T) = N(w, T)$ and $d(v, T) = 3$.*

Lemma 2.4 ([8]). *Let $P = uvwx$ be a path and T a triangle of G such that T is disjoint from P. Then the following hold:*

(a) *If $\sum_{y \in V(P)} d(y, T) \geq 9$, then $G[V(P \cup T) - z]$ contains two disjoint triangles for some $z \in V(P)$. Furthermore, if $G[V(P \cup T) - y]$ does not contain two disjoint triangles for every $y \in \{u, x\}$, then $d(z, T) = 0$.*

(b) *If $\sum_{y \in V(P)} d(y, T) \geq 8$, then $G[V(P \cup T)]$ contains two disjoint cycles.*

Lemma 2.5 ([8]). *Let s and t be two integers with $t \geq s \geq 3$ and $t \geq 4$. Let C_1 and C_2 be two disjoint cycles of G with lengths s and t, respectively. Suppose that $\sum_{x \in V(C_2)} d(x, C_1) \geq 2t + 1$. Then $G[V(C_1 \cup C_2)]$ contains two disjoint cycles C' and C'' such that $l(C') + l(C'') < s + t$.*

3. Proof of Theorem 1.3

Let k be a positive integer and G a graph of order $n \geq 3k$. Assume that $d(x) + d(y) \geq 4k$ for every pair of vertices x and y of distance 2 in G. Suppose to the contrary, that G does not contain k disjoint cycles and G does not belong to $F_{3,k,n}$. Let t be the greatest integer in $\{1, 2, \ldots, k\}$ such that G contains t disjoint cycles. Then $t < k$. Also $t \geq 1$, since the degree condition implies G has a vertex with degree at least 2. We may choose t disjoint cycles $C_1, C_2, \ldots, C_t$ such that

$$\sum_{i=1}^{t} l(C_i) \text{ is minimum.} \quad (1)$$