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On line graphs of subcubic triangle-free graphs

Andrea Munaro

Laboratoire G-SCOP, Univ. Grenoble Alpes, France

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ABSTRACT

Line graphs constitute a rich and well-studied class of graphs. In this paper, we focus on three different topics related to line graphs of subcubic triangle-free graphs. First, we show that any such graph G has an independent set of size at least $3|V(G)|/10$, the bound being sharp. As an immediate consequence, we have that any subcubic triangle-free graph G, with n_i vertices of degree i, has a matching of size at least $3n_1/20 + 3n_2/10 + 9n_3/20$. Then we provide several approximate min-max theorems relating cycle-transversals and cycle-packings of line graphs of subcubic triangle-free graphs. This enables us to prove Jones' Conjecture for claw-free graphs with maximum degree 4. Finally, we concentrate on the computational complexity of Feedback Vertex Set, Hamiltonian Cycle and Hamiltonian Path for subclasses of line graphs of subcubic triangle-free graphs.

1. Introduction

The *line graph* $L(G)$ of a graph G is the graph having as vertices the edges of G, two vertices being adjacent if the corresponding edges intersect. Line graphs constitute a rich and well-studied class of graphs. In this paper, we concentrate on the subclass of line graphs of subcubic triangle-free graphs (a *subcubic* graph is a graph with maximum degree at most 3). It is easy to see that such graphs have maximum degree at most 4. In Section 2, we provide several characterizations of this class. In particular, we observe that the class of line graphs of subcubic triangle-free graphs coincides with the class of (K_4, claw, diamond)-free graphs. Moreover, we show that the line graphs of cubic triangle-free graphs are exactly those 4-regular graphs for which every edge belongs to exactly one K_3.

In Section 3, we consider the *independence number* $\alpha(G)$ of a graph G, the maximum size of an independent set of G. The famous Brooks' Theorem asserts that every connected graph G which is neither a complete graph nor an odd cycle must be $\Delta(G)$-colourable, and so $\alpha(G) \geq |V(G)|/\Delta(G)$. Following this result, several authors considered the problem of finding tight lower bounds for the independence number of graphs having bounded maximum degree and not containing cliques on 3 or 4 vertices [18,19,25,34,42]. Kang et al. [29] showed that if G is a connected (K_4, claw)-free 4-regular graph on n vertices then, apart from three exceptions, $\alpha(G) \geq (8n-3)/27$. Motivated by this result, we show that if G is a (K_4, claw, diamond)-free graph on n vertices, then $\alpha(G) \geq 3n/10$. This gives a tight bound, as can be seen by considering the following:

Example 1. Let G be the graph obtained from a 7-cycle C_7, where $V(C_7) = \{v_1, \ldots, v_7\}$, by adding the edges v_2v_5, v_3v_6 and v_4v_7. Clearly, we have $\alpha(L(G)) = \alpha'(G) = 3$, where the *matching number* $\alpha'(G)$ is the size of a maximum matching of G.

The well-known Petersen's Theorem asserts that every 3-regular bridgeless graph has a perfect matching and the question of whether a graph admits a perfect matching has been deeply investigated (see [1] for a survey). On the other hand, not much is known about general lower bounds for the matching number. Biedl et al. [8] showed that every subcubic graph G has a matching of size $(|V(G)|-1)/3$ and that every cubic graph G has a matching of size $(4|V(G)|-1)/9$. Henning et al. [26]

E-mail address: Andrea.Munaro@grenoble-inp.fr.

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investigated lower bounds in the case of cubic graphs with odd girth. In particular, they showed that every connected cubic triangle-free graph G has a matching of size $(11|V(G)| - 2)/24$. By recalling that the matchings of a graph G are in bijection with the independent sets of its line graph $L(G)$, our result on the independence number of (K_4, claw, diamond)-free graphs directly translates into a tight lower bound for the matching number. Indeed, we show that if G is a subcubic triangle-free graph with n_i vertices of degree i, then $\alpha'(G) \geq 3n_1/20 + 3n_2/10 + 9n_3/20$.

Consider now a family of graphs $\mathcal{F}$. An *$\mathcal{F}$-transversal* of a graph G is a subset $T \subseteq V(G)$ such that T intersects all the subgraphs of G isomorphic to a graph in $\mathcal{F}$. An *$\mathcal{F}$-packing* of G is a set of vertex-disjoint subgraphs of G, each isomorphic to a graph in $\mathcal{F}$. In Section 4, we are interested in $\mathcal{F}$-transversals and $\mathcal{F}$-packings, where $\mathcal{F}$ is the family of cycles or the family of graphs isomorphic to K_3. Therefore, we talk about *cycle-transversals* and *triangle-transversals*, with the obvious meaning. Note that, in the literature, a cycle-transversal is also known as a *feedback vertex set*. We denote by $\tau_\Delta(G)$ (resp. $\tau_c(G)$) the minimum size of a triangle-transversal (resp. feedback vertex set) of G and by $\nu_\Delta(G)$ (resp. $\nu_c(G)$) the maximum number of vertex-disjoint triangles (resp. cycles) of G.

$\mathcal{F}$-transversals and $\mathcal{F}$-packings are related in an obvious way. Indeed, each transversal must contain at least one vertex for each subgraph in a packing and so $\tau_c(G) \geq \nu_c(G)$ and $\tau_\Delta(G) \geq \nu_\Delta(G)$. A natural question is whether $\tau_c(G)$ and $\tau_\Delta(G)$ can be upper bounded in terms of $\nu_c(G)$ and $\nu_\Delta(G)$. It turns out we have a positive answer in both cases. Indeed, by taking the vertex set of $\nu_\Delta(G)$ vertex-disjoint triangles, we have that $\tau_\Delta(G) \leq 3\nu_\Delta(G)$ (it is easy to see that equality holds for any K_{3k+2} with $k \geq 1$). The case of cycles was addressed in a seminal work by Erdős and Pósa [15] showing the sharp bound $\tau_c(G) = O(\nu_c(G) \log \nu_c(G))$. These results are examples of the so-called min-max theorems: one parameter is characterized by its obstructing analogue, or dual. Indeed, either G contains k vertex-disjoint triangles (resp. cycles) or it contains a triangle-transversal (resp. cycle-transversal) of size $3k$ (resp. $O(k \log k)$).

Kloks et al. [30] conjectured that the bound given by Erdős and Pósa can be greatly improved in the case of planar graphs:

Conjecture 2 (*Kloks et al. [30]*)**.** *If G is a planar graph, then $\tau_c(G) \leq 2\nu_c(G)$.*

Conjecture 2 is also known as Jones' Conjecture. If true, it would be sharp, as can be seen by considering wheel graphs. Kloks et al. [30] showed it holds for outerplanar graphs and, in general, they proved the weaker $\tau_c(G) \leq 5\nu_c(G)$. Subsequently and independently, the factor 5 was replaced by 3 in a series of papers [9,12,35]. To the best of our knowledge, Conjecture 2 is open even in the case of subcubic graphs.

In Section 4, we observe that a triangle-transversal of $G = L(H)$, where H is a subcubic triangle-free graph, essentially corresponds to an edge cover of H. This enables us to show the sharp bound $\tau_\Delta(G) \leq 3\nu_\Delta(G)/2$. It is easy to see that line graphs of subcubic triangle-free graphs are not necessarily planar. Nevertheless, we show that, for any such graph G, the sharp bound $\tau_c(G) \leq 2\nu_c(G)$ holds. Using this result, we can finally verify Conjecture 2 for claw-free graphs with maximum degree at most 4.

Feedback Vertex Set is the problem of deciding, given a graph G and an integer k, whether $\tau_c(G) \leq k$. Ueno et al. [43] showed that Feedback Vertex Set can be solved in polynomial time for graphs with maximum degree 3 by a reduction to a matroid parity problem. On the other hand, Feedback Vertex Set becomes NP-hard for graphs with maximum degree 4, even if restricted to be planar, as shown by Speckenmeyer [40] (see also [41]). In Section 5, we strengthen this result by showing the NP-hardness for line graphs of planar cubic bipartite graphs. This is done in two steps. We first show that if G is the line graph of a cubic triangle-free graph H, then $\tau_c(G) \geq |V(G)|/3 + 1$, with equality if and only if H contains a Hamiltonian path. We then show that the well-known Hamiltonian Path is NP-complete even for planar cubic bipartite graphs. This matches the fact that Hamiltonian Cycle is NP-complete for planar cubic bipartite graphs [2] and may be of independent interest. We conclude the section with an inapproximability result for Feedback Vertex Set restricted to line graphs of subcubic triangle-free graphs.

Despite the fact that Hamiltonicity in line graphs has been widely investigated, beginning with the works of Chartrand [10,11] and Harary and Nash-Williams [23], to the best of our knowledge no result is known about Hamiltonian Cycle for line graphs. Concerning Hamiltonian Path, Bertossi [7] showed that the problem is NP-complete for line graphs. Lai and Wei [33] strengthened this result by showing that it remains NP-hard even when restricted to line graphs of bipartite graphs. In Section 6, we prove that Hamiltonian Cycle remains NP-hard for line graphs of 1-subdivisions of planar cubic bipartite graphs and for line graphs of planar cubic bipartite graphs. Finally, we show that Hamiltonian Path remains NP-hard for line graphs of 1-subdivisions of planar cubic bipartite graphs, thus strengthening the result by Lai and Wei [33].

As a side remark, let us mention that line graphs of subcubic triangle-free graphs are not necessarily 3-colourable, as Example 1 shows. Moreover, if G is the line graph of a cubic triangle-free graph H, then G is 3-colourable if and only if H is of class 1. This implies that 3-Colourability is NP-complete even when restricted to line graphs of cubic triangle-free graphs [31].

We assume the reader is familiar with notions of graph theory; for those not defined here, we refer to [44]. Note that we consider only finite and undirected graphs with no loops and no multiple edges. A *k-vertex* is a vertex of degree k. A *cubic* graph is a 3-regular graph. The complete graph on n vertices is denoted by K_n and the complete bipartite graph with partition classes of size n and m is denoted by $K_{n,m}$. A *triangle* is (a graph isomorphic to) K_3, a *claw* is $K_{1,3}$ and a *diamond* is the graph obtained from K_4 by removing an edge. A 1-*subdivision* of G is the graph obtained from G by adding a new vertex for each edge of G, i.e. each edge is replaced by a path of length 2.