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Cycle extension in edge-colored complete graphs

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ABSTRACT

Let G be an edge-colored graph. The minimum color degree of G is the minimum number of different colors appearing on the edges incident with the vertices of G. In this paper, we study the existence of properly edge-colored cycles in (not necessarily properly) edge-colored complete graphs. Fujita and Magnant (2011) conjectured that in an edge-colored complete graph on n vertices with minimum color degree at least $(n+1)/2$, each vertex is contained in a properly edge-colored cycle of length k, for all k with $3 \leq k \leq n$. They confirmed the conjecture for $k = 3$ and $k = 4$, and they showed that each vertex is contained in a properly edge-colored cycle of length at least 5 when $n \geq 13$, but even the existence of properly edge-colored Hamilton cycles is unknown (in complete graphs that satisfy the conditions of the conjecture). We prove a cycle extension result that implies that each vertex is contained in a properly edge-colored cycle of length at least the minimum color degree.

1. Introduction

We only consider finite and simple graphs. For terminology and notation not defined here, we refer the reader to [3].

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. An *edge-coloring* of G is a mapping $col : E(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. A graph G is called an *edge-colored graph* (or throughout this paper simply a *colored graph*) if its edge set is assigned an edge-coloring. We say that a colored graph G is a *properly colored graph* (or *PC graph* for short) if each pair of adjacent edges (i.e., edges that have precisely one end vertex in common) in G are assigned distinct colors. For a vertex v in a colored graph G, the *color degree* of v, denoted by $d^c_G(v)$, is the number of distinct colors appearing on the edges incident with v, and $\delta^c(G)$ denotes the minimum color degree of G taken over all vertices of G.

For a colored graph G, the color of an edge $e \in E(G)$ is denoted by $col_G(e)$. For a subgraph H of G, we use $col_G(H)$ to denote the set of colors appearing on the edges of H. For vertex-disjoint subgraphs F and H of G, we use $col_G(F, H)$ to denote the set of colors appearing on the edges between F and H. If F contains only one vertex v, then we write $col_G(v, H)$ instead of $col_G(\{v\}, H)$. When there is no ambiguity, we often write $d^c(v)$ for $d^c_G(v)$, $col(e)$ for $col_G(e)$, $col(H)$ for $col_G(H)$, $col(F, H)$ for $col_G(F, H)$ and $col(v, H)$ for $col_G(v, H)$. For a color $i \in col(G)$, we use G^i to denote the spanning subgraph of G induced by the edges of color i. We denote by $\Delta^{mon}(G)$ the *maximum monochromatic degree* of G, i.e., $\Delta^{mon}(G) = \max\{\Delta(G^i) : i \in col(G)\}$. We say a path (cycle) has length k if the path (cycle) contains exactly k edges. We use $|C|$ to denote the length (and hence the order) of the cycle C.

In this paper, we study the existence of PC cycles in colored complete graphs. This topic has been well-studied. We refer to Chapter 16 in [1] for a survey. In this field, maximum monochromatic degree conditions and color degree conditions

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for guaranteeing the existence of certain cycles have often been considered. Regarding maximum monochromatic degree conditions, Bollobás and Erdős [2] already conjectured back in the 1970s that every colored K_n contains a PC Hamilton cycle if $\Delta^{mon}(K_n) < \lfloor \frac{n}{2} \rfloor$. Very recently, Lo [9] confirmed this conjecture asymptotically. When investigating long PC cycles, Wang et al. [10] proved that if $\Delta^{mon}(K_n) < \lfloor \frac{n}{2} \rfloor$, then K_n contains a PC cycle of length at least $\lceil \frac{n}{2} \rceil + 2$. Color degree conditions have been studied in the context of the existence of short PC cycles [5,6] as well as long PC cycles [7,8] in general (not necessarily complete) colored graphs.

Our main motivation for the results in this paper is the following conjecture due to Fujita and Magnant [4].

Conjecture 1 (*Fujita and Magnant [4]*). *Let G be a colored K_n. If $\delta^c(G) \geq \frac{n+1}{2}$, then each vertex of G is contained in a PC cycle of length k, for all k with $3 \leq k \leq n$.*

In the same paper, they presented a class of colored complete graphs to show that the statement of Conjecture 1 would be best possible (the lower bound on $\delta^c(G)$ cannot be improved), and they established results on short PC cycles to support their conjecture.

Theorem 2 (*Fujita and Magnant [4]*). *Let G be a colored K_n with $\delta^c(G) \geq \frac{n+1}{2}$. Then each vertex of G is contained in PC cycles of length 3 and 4. Moreover, if $n \geq 13$, then each vertex of G is contained in a PC cycle of length at least 5.*

For other results related to Conjecture 1, we recommend Lo's papers [7] and [9]. In [7], it is proved that a colored graph G contains a PC cycle of length at least $\delta^c(G) + 1$ when $\delta^c(G) \geq \frac{n+1}{2}$. In [9], the main result implies that a colored complete graph G (of order n with n sufficiently large) contains a PC Hamilton cycle when $\delta^c(G) \geq (1/2 + \epsilon)n$ for any arbitrarily small constant $\epsilon > 0$. In this context, we focus on the following problem.

Problem 1. Let G be a colored K_n with $\delta^c(G) \geq \frac{n+1}{2}$. Determine the largest value of $f(n)$ such that each vertex of G is contained in a PC cycle of length at least $f(n)$.

The existence of a PC Hamilton cycle would solve the problem and show that $f(n) = n$. By Theorem 2, $f(n) \geq 4$, and $f(n) \geq 5$ when $n \geq 13$. In this paper, we first present a PC cycle extension theorem, and then show that it implies that $f(n) \geq \delta^c(G)$.

Theorem 3. *Let G be a colored K_n, and let C be a PC cycle of length k in G. If $\delta^c(G) \geq \max\{\frac{n-k}{2}, k\} + 1$, then G contains a PC cycle C^* such that $V(C) \subset V(C^*)$ and $|C^*| > k$.*

We postpone the proof of Theorem 3 to Section 3. From Theorem 3, we can obtain the following corollaries.

Corollary 4. *Let G be a colored K_n with $\delta^c(G) \geq \frac{n+1}{2}$. Then each vertex of G is contained in a PC cycle of length at least $\delta^c(G)$, i.e., $f(n) \geq \delta^c(G)$.*

Proof. Let G be a colored K_n with $\delta^c(G) \geq \frac{n+1}{2}$. By Theorem 2, each vertex of G is contained in a PC cycle. For an arbitrary vertex $v \in V(G)$, denote by C the longest PC cycle containing v. We will prove that $|C| \geq \delta^c(G)$. Suppose the contrary. Then $3 \leq |C| \leq \delta^c(G) - 1$, and $\delta^c(G) \geq \frac{n+1}{2} > \frac{n-|C|}{2} + 1$. This implies that $\delta^c(G) \geq \max\{\frac{n-|C|}{2}, |C|\} + 1$. By Theorem 3, there exists a longer PC cycle containing $V(C)$, a contradiction. □

Corollary 5. *Let G be a colored K_n, and let C be a PC cycle of length k in G. If $k \geq \frac{n}{3}$, then there exists a PC cycle C^* of length at least $\delta^c(G)$ such that $V(C) \subseteq V(C^*)$.*

Proof. Let G be a colored K_n, let C be a PC cycle of length $k \geq \frac{n}{3}$ in G, and let C^* be a longest PC cycle in G with $V(C) \subseteq V(C^*)$. Then, clearly $|C^*| \geq k \geq \frac{n}{3}$. We will prove that $|C^*| \geq \delta^c(G)$. Suppose the contrary. Then, $\delta^c(G) \geq |C^*| + 1 = \max\{\frac{n-|C^*|}{2}, |C^*|\} + 1$. By Theorem 3, there exists a longer PC cycle containing $V(C^*)$, a contradiction. □

In the light of the results we have obtained, we propose the following two problems. We first note that, by a result due to Yeo in [11], every colored K_n with $\delta^c(K_n) \geq 2$ contains a PC cycle (in fact, a PC C_3 or a PC C_4).

Problem 2. Does every colored K_n with $\delta^c(K_n) \geq 2$ contain a PC cycle of length at least $\delta^c(K_n)$?

Problem 3. Let C be a PC cycle in a colored K_n. Does there exist a PC cycle C' with $|C'| \geq \delta^c(K_n)$ and $V(C) \subseteq V(C')$?

Note that in Problem 3, C' is not necessarily distinct from C.

In order to present our proof of Theorem 3 in Section 3, we need some additional terminology and notation, and we prove two auxiliary lemmas in the next section.