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Rainbow Hamilton cycles and lopsidependency

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ABSTRACT

The Lovász Local Lemma is a powerful probabilistic tool used to prove the existence of combinatorial structures which avoid a set of constraints. A standard way to apply the local lemma is to prove that the set of constraints satisfy a lopsidependency condition and obtain a lopsidependency graph. For instance, Erdős and Spencer used this framework to posit the existence of Latin transversals in matrices provided no symbol appears too often in the matrix.

The local lemma has been used in various ways to infer the existence of rainbow Hamilton cycles in complete graphs when each colour is used at most $O(n)$ times. However, the existence of a lopsidependency graph for Hamilton cycles has neither been proved nor refuted. All previous approaches have had to prove a variant of the local lemma or reduce the problem of finding Hamilton cycles to finding another combinatorial structure, such as Latin transversals. In this paper, we revisit the question of whether or not Hamilton cycles have a lopsidependency graph and give a positive answer for this question. We also use the resampling oracle framework of Harvey and Vondrák to give a polynomial time algorithm for finding rainbow Hamilton cycles in complete graphs.

1. Introduction

In combinatorics, the Lovász Local Lemma (LLL) is a very powerful probabilistic tool which has seen many applications (for some classic examples, see [4,11]). The original LLL was only applicable to probability spaces where the events formed a "dependency graph". This was later extended to the setting of "lopsidependency graphs" by Erdős and Spencer [12]. A similar extension of the LLL was independently obtained by Albert, Frieze, and Reed in their work on "rainbow Hamilton cycles" [3]. Lu, Mohr, and Székely have undertaken a study of probability spaces and events that have a lopsidependency graph [24]. Some examples from their work include random matchings in complete uniform hypergraphs, random spanning trees in complete graphs, and random permutations. Although Albert, Frieze, and Reed did apply the LLL to random Hamilton cycles in complete graphs, they did not show that this scenario leads to a lopsidependency graph. To our knowledge, that statement is neither proven nor refuted by any result appearing in the literature. We prove indeed random Hamilton cycles do lead to a lopsidependency graph, thereby extending the list of examples collected in the survey of Lu, Mohr, and Szekely.

Over the past few years, there has also been much work on algorithmic forms of the LLL, even for settings involving lopsidependency graphs [1,2,17–20,22,27]. Harvey and Vondrák [20] define an abstract notion of resampling oracles, and show that the LLL has an algorithmic proof in any scenario with resampling oracles. They also show that the existence of resampling oracles implies that the scenario involves a lopsidependency graph. We design efficient resampling oracles for the scenario of random Hamilton cycles in complete graphs, implying that this scenario involves a lopsidependency graph.

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Finally, we discuss a recent enhancement of the LLL known as the cluster expansion criterion. This gives sharper results in several applications of the LLL. We use this criterion in the scenario of random Hamilton cycles to give new results on rainbow Hamilton cycles that slightly strengthen those of Albert, Frieze, and Reed. Furthermore our results are algorithmic due to the framework of Harvey and Vondrák and our efficient resampling oracles.

1.1. Background

A cycle in a graph is called a *Hamilton* cycle if every vertex appears exactly once. If the graph is edge-coloured then the Hamilton cycle is called *rainbow* if distinct edges are assigned distinct colours. Define the function $k(n)$ to be the minimum value that satisfies the following condition. No matter how we edge-colour the complete graph K_n, if every colour appears at most $k(n)$ times then there exists a rainbow Hamilton cycle.

If we pick a vertex v and assign the same colour to all edges incident to v then this graph does not contain a rainbow Hamilton cycle. So an easy upper bound is $k(n) < n - 1$. Hahn and Thomassen [16] conjectured that this is essentially tight. More precisely, they conjectured that for some constant $\gamma > 0$ and any n sufficiently large, we have $k(n) \geq \gamma n$.

The earliest result in this direction is due to Hahn and Thomassen [16] who proved that $k(n) = \Omega\left(n^{1/3}\right)$. Frieze and Reed [15] improved this to $k(n) = \Omega\left(\frac{n}{\log n}\right)$. Finally, Albert, Frieze, and Reed [3] closed the gap.

Theorem 1 (*Albert, Frieze, Reed [3]*). *Let $\gamma < 1/64$.*[1] *There exists $n_0 = n_0(\gamma)$ such that if $n \geq n_0$ then the following holds. For any edge-colouring of K_n, if any colour appears on at most γn edges then K_n contains a rainbow Hamilton cycle.*

Other related works. The present work considers the existence of a rainbow Hamilton cycle under an adversarial colouring in the complete graph. It is also interesting to ask when rainbow Hamilton cycles exist under different settings. The existence of rainbow Hamilton cycles in the Erdős–Renyi random graph model and a uniform random colouring was studied by [9,13,14]. Let $G_{n,p,c}$ be the random graph on n vertices where each edge is included with probability p and each edge receives one of c colours uniformly at random. Ferber and Krivelevich [13] show that, w.h.p.,[2] the random graph $G_{n,p,c}$ contains a rainbow Hamilton cycle as long as $c \geq (1+\varepsilon)n$ and $p \geq \frac{\log n + \log\log n + \omega(1)}{n}$. This result is tight as $c \geq n$ colours are required and it is well-known that the threshold of p is required just to have Hamiltonicity [23]. Generalizations of this result to hypergraphs have also appeared in the literature (see [6,13]).

There are also some results for other graph models. Janson and Wormald [21] showed that a random d-regular graph with a random colouring where each colour appears $d/2$ times ($d \geq 8$ is even) has a rainbow Hamilton cycle w.h.p. Bal et al. [5] study the existence of rainbow Hamilton cycles in a random geometric graphs where each edge is given a uniform colour from a set of $\Theta(n)$ colours. They show that, w.h.p., a rainbow Hamilton cycle "emerges" as soon as the minimum degree of the graph is at least 2.

1.2. The Lovász Local Lemma

We first review some results related to the Lovász Local Lemma.

Definition 2. Let Ω be a probability space and $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a collection of "bad" events from Ω. Let G be a graph with vertex set $[n] = \{1, \ldots, n\}$ and edge set $E \subseteq \binom{[n]}{2}$. Denote with $\Gamma(i)$ the neighbourhood of i and $\Gamma^+(i) = \Gamma(i) \cup \{i\}$. We say that G is a *dependency* graph for $\mathcal{F}$ if for all $i \in [n]$ and $J \subseteq [n] \setminus \Gamma^+(i)$

$$\Pr\left[F_i \mid \cap_{j\in J}\overline{F_j}\right] = \Pr\left[F_i\right]. \tag{1}$$

If, instead, (1) is replaced by

$$\Pr\left[F_i \mid \cap_{j\in J}\overline{F_j}\right] \leq \Pr\left[F_i\right] \tag{2}$$

then G is a *lopsidependency* graph for $\mathcal{F}$.

Remark 3. Observe that if G is a dependency (resp. lopsidependency) graph for the events $\{F_i\}_{i\in[n]}$ and $I \subseteq [n]$ then the vertex-induced subgraph $G[I]$ is a dependency (resp. lopsidependency) graph for the events $\{F_i\}_{i\in I}$. Indeed, since $I \subseteq [n]$, if (1) (resp. (2)) holds for $\{F_i\}_{i\in[n]}$ then (1) (resp. (2)) holds for $\{F_i\}_{i\in I}$.

Theorem 4 (*Lovász Local Lemma [11,12,29]*). *Let $F_1, \ldots, F_n$ be a set of events with associated lopsidependency graph G. Suppose there exists $x_1, \ldots, x_n \in [0, 1)$ such that for all $i \in [n]$*

$$\Pr\left[F_i\right] \leq x_i \prod_{j\in\Gamma(i)} \left(1 - x_j\right).$$

[1] The original paper claimed that $\gamma < 1/32$. This was later corrected to $\gamma < 1/64$. The comment can be found at http://www.combinatorics.org/ojs/index.php/eljc/article/view/v2i1r10/comment.

[2] A sequence of events E_n is said to occur with high probability (w.h.p.) if $\lim_{n\to\infty} \Pr[E_n] = 1$.