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On the number of vertices of positively curved planar graphs

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ABSTRACT

For a connected simple graph embedded into a 2-sphere, we show that the number of vertices of the graph is less than or equal to 380 if the degree of each vertex is at least three, the combinatorial vertex curvature is positive everywhere, and the graph is different from prisms and antiprisms. This gives a new upper bound for the constant brought up by DeVos and Mohar in their paper from 2007. We also show that if a graph is embedded into a projective plane instead of a 2-sphere but satisfies the other properties listed above, then the number of vertices is at most 190.

1. Introduction

Let G be a connected simple (finite or infinite) graph with the vertex set $V := V(G)$ and the edge set $E := E(G)$, and suppose that G is embedded locally finitely into a 2-dimensional topological manifold Ω without boundary. Each component of $\Omega \setminus G$ will be called an open face, or just a face, of G, and we assume that each face of G is homeomorphic to an open disk. Thus for example a finite planar graph should be embedded into a 2-sphere instead of the Euclidean plane. The set of faces of G is denoted by $F := F(G)$. Throughout the paper we always assume that, in addition to the properties above, G satisfies the following properties: $3 \le \deg x < \infty$ for every $x \in V$ and $3 \le |\sigma| < \infty$ for every $\sigma \in F$, where $\deg x$ denotes the *degree* of x and indicates the number of edges incident with x, and $|\sigma|$ denotes the *size (degree, girth)* of σ and means the number of edges bounding σ.

For each $x \in V$ we define the *combinatorial (vertex) curvature* ϕ at x by

$$\phi(x) = 1 - \frac{\deg x}{2} + \sum_{\sigma : x \in V(\sigma)} \frac{1}{|\sigma|}.$$

Here $V(\sigma)$ denotes the set of vertices incident to σ, hence the sum is taken over all the faces containing x on their boundaries. However, the reciprocal of the size of each face should be added the number of times that x is incident to the face, so a face can appear more than once in the above formula.

Combinatorial curvature can be considered a discrete version of sectional curvature on Riemannian manifolds, and in fact it is related to the atomic curvature on the underlying surface. See for instance [3]. It is told in some literature that combinatorial curvature was introduced by Gromov [4], but a similar concept already came out in a book of Nevanlinna published about eight decades ago [8, Chap. XII]. One can check that the characteristic number called *excess* in [8] (cf. [10, Def. 6.2]) is the same as $2\phi(x)$.

It is expected, as in the continuous case, that studying combinatorial curvature could reveal various geometric properties of the graph. For example, Higuchi proved in [5] that if $\phi(x) < 0$ for every $x \in V$, then G satisfies a strong isoperimetric

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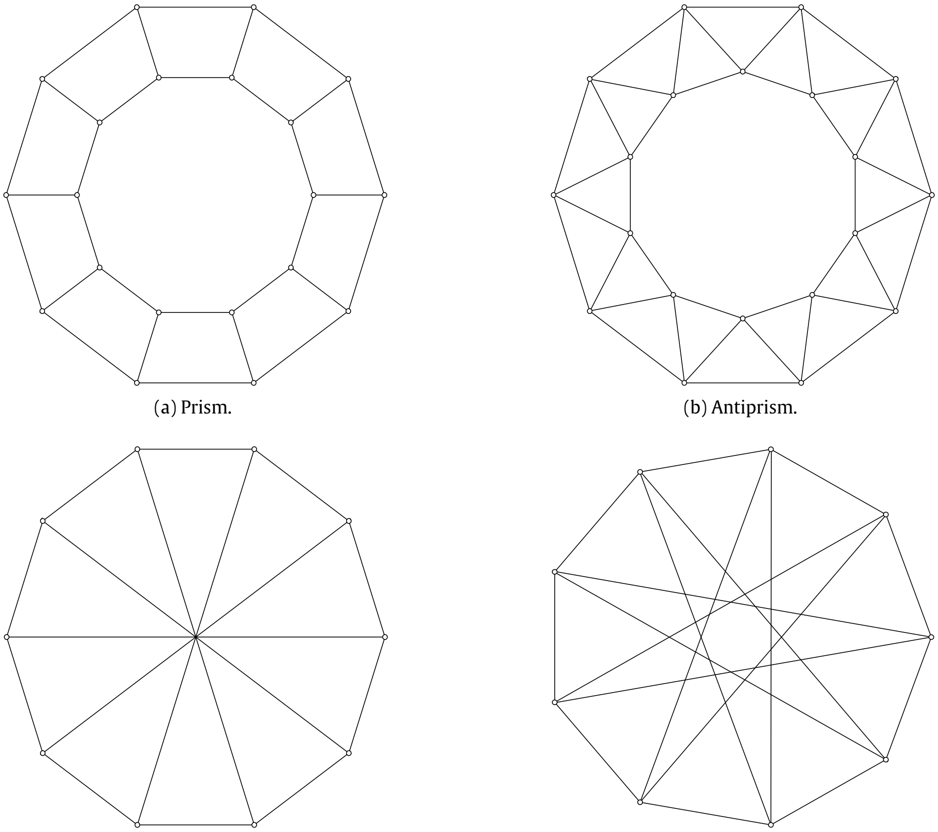

(a) Prism. (b) Antiprism.

(c) Projective plane prism(there is no vertex at the center). (d) Projective plane antiprism.

Fig. 1. Examples of a prism, an antiprism, and their projective plane analogues.

inequality. Some similar results can be found in the works of other researchers, for example in [11,12,15,17]. On the other hand, one can ask what would happen if G is *positively curved*; i.e., if $\phi(x) > 0$ for every $x \in V$. Higuchi conjectured in the same paper [5] the following statement, as a discrete analogue of Myer's theorem [7].

Higuchi's Conjecture. Suppose G is a connected simple graph embedded into a 2-sphere such that the degree of each vertex of G is at least three. If G is positively curved, then G is finite.

Higuchi's conjecture was verified by Sun and Yu in [14] for cubic planar graphs, and it is fully resolved by DeVos and Mohar in [3]. In fact, DeVos and Mohar proved the following theorem.

Theorem 1 (*DeVos and Mohar, [3]*)**.** *Suppose G is a connected simple graph embedded into a 2-dimensional topological manifold Ω without boundary, and suppose that the degree of each vertex of G is at least three. If G is positively curved, then it is finite and Ω is homeomorphic to either a 2-sphere or a projective plane. Moreover, if G is not a prism, an antiprism, or one of their projective plane analogues, then $n(V) \leq 3444$.*

The notation $n(\cdot)$ denotes the cardinality of the given set. Examples of a prism, an antiprism, and their projective plane analogues are shown in Fig. 1. Also note that prisms, antiprisms, and their projective plane analogues are denoted by A_n, B_n, and P_n, respectively, in [2] and [16].

DeVos and Mohar mentioned in the same paper [3] that it would be interesting to find the smallest constant, say C_1, which could take the place of 3444 in Theorem 1. Similarly let C_2 be the smallest such constant, but in this case only graphs embedded into projective planes are considered.

Though we know some upper and lower bounds for C_1 and C_2, the exact values of them have not been answered yet. As indicated in [3] the skeleton of the great rhombic icosidodecahedron is positively curved and has 120 vertices. Réti, Bitay, and Kosztolányi constructed in [13] a positively curved graph with 138 vertices, embedded into a 2-sphere, and different from prisms and antiprisms. Later in [9] Nicholson and Sneddon gave examples of positively curved graphs with 208 and