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Stochastic tensors and approximate symmetry

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ABSTRACT

Triply stochastic cubic tensors, or sharply transitive sets of doubly stochastic matrices, are decompositions of the all-ones matrix as the sum of an ordered set of bistochastic matrices. They combine to yield so-called weak approximate quasigroups and Latin squares. Approximate symmetry is implemented by the stochastic matrix actions of quasigroups on homogeneous spaces, thereby extending the concept of exact symmetry as implemented by permutation matrix actions of groups on coset spaces. Now approximate quasigroups and Latin squares are described as being strong if they occur within quasigroup actions. We study these weak and strong objects, in particular examining the location of the latter within the polytope of triply stochastic cubic tensors. We also establish the rudiments of an algebraic structure theory for approximate quasigroups. Upon relaxation from probability distributions to their supports, approximate quasigroups furnish non-associative analogues of (set-theoretical) hypergroups.

1. Introduction

1.1. Latin squares and triply stochastic cubic tensors

A *Latin square* on an ordered n-element set $X = \{x_1 < \cdots < x_n\}$ is defined as an $n \times n$ square matrix, containing n copies of each element of X among its entries, such that each element appears precisely once in each row and each column of the square. While the definition is combinatorial, it may be brought into the purview of linear algebra on noting that the occurrences of each element x_i of X in the square form a permutation matrix, so each Latin square on X is specified by the ordered stack of respective permutation matrices for $x_1, \dots, x_n$, the sum of these matrices being the square all-ones matrix J_n of degree n. This approach was the basis for a formal enumeration of the Latin squares of order n [20, p. 294].

Now just as doubly stochastic (or bistochastic) square matrices serve to extend the concept of a permutation matrix, one may define a *triply stochastic* or *tristochastic* cubic tensor, or (*weak*) *approximate Latin square*, of degree n to be an ordered list or stack $(S_1, \dots, S_n)$ of doubly stochastic matrices of degree n, such that the sum of the matrices $S_1, \dots, S_n$ is J_n (Definition 3.4). These objects form one of the two main themes of the paper.

1.2. Quasigroups and approximate symmetry

Each Latin square on an ordered n-element set X may be considered algebraically, as the body of the multiplication table of a *quasigroup* $(X, \cdot)$ for which the respective rows and columns are labeled by the elements of X in order. The Latin square property means that for any two elements x, y, z in X, the equation $x \cdot y = z$ specifies the third element uniquely. (One writes

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$x = z/y$ and $y = x\backslash z$.) Groups have this cancellation property, so the multiplication table of any group is a Latin square, and groups form a special, associative, case of quasigroups. The importance of groups stems from their role in the analysis of exact symmetry. In particular, any transitive permutation action of a group Q is similar to the action of the group on the space $P\backslash Q$ of cosets of a point stabilizer P, a subgroup of Q. More generally, for a subquasigroup P of a finite quasigroup Q, one may similarly define an action of Q on a coset or *homogeneous space* $P\backslash Q$ (Section 2.2). In general, the action of elements of Q on $P\backslash Q$ is by Markov matrices (2.4) (compare [21], [22, Ch. 4], and the introduction to [12]). The Markov matrices reduce to permutation matrices if Q is a group. Thus linear algebra affords an exact theory of approximate symmetry provided by quasigroup actions, extending the combinatorial exact symmetry provided by group actions. This approach to approximate symmetry forms the second main theme of the paper.

1.3. Strong approximate quasigroups

In the context where the doubly stochastic square matrices that combine to a triply stochastic cubic tensor happen to be permutation matrices, it is said that the set of corresponding permutations is *sharply transitive* [19]. (In Baer's original paper on the subject, sharp transitivity was called "simple transitivity" [1]. On the other hand, Cameron used the term "uniform transitivity", placing sharp transitivity into the context of matroids and permutation geometries [6].) If y is an element of a quasigroup $(X, \cdot)$ on the ordered set X as above, define the *right multiplication* $R_Q(y) : X \to X; x \mapsto x \cdot y$. Then $\big(R_Q(x_1), \ldots, R_Q(x_n)\big)$ is a sharply transitive set of permutations of X. This process may be reversed to obtain a quasigroup from a sharply transitive set of permutations.

Now extending from exact symmetry (i.e., permutation matrices) to approximate symmetry, we are led to consider sets of quasigroup action matrices summing to J_n. These sets are again described as being *sharply transitive* [12]. Extending the correspondence between sharply transitive sets of permutations and (exact) quasigroups, these sharply transitive sets of quasigroup action matrices yield *strong approximate quasigroups* (Definitions 2.7 and 2.12). In particular, it is shown that exact quasigroups are strongly approximate quasigroups (Theorem 2.13).

With our current state of knowledge, it is hard to construct strong approximate quasigroups. One available technique (reviewed in Section 4) was described in the earlier paper [12], using quasigroup permutation representations of 12-element quasigroups $Q(\Gamma)$ obtained by twisting the multiplication table of the group $S_3 \times \mathbb{Z}/_2$ at locations determined by a directed graph Γ on the vertex set S_3. Various examples of strong approximate quasigroups that may be produced with this technique are exhibited in Section 5, along with a smaller example obtained by an *ad hoc* method (Section 5.1).

1.4. Parastrophy and weak approximate quasigroups

There are two ways that a triply stochastic cubic $(0, 1)$-tensor (sharply transitive set of permutation matrices) corresponds with a quasigroup or Latin square. On the one hand, one may take the occurrences of each symbol in the Latin square (the body of the quasigroup multiplication table), as described in Section 1.1. On the other hand, one has the permutations given by the respective right multiplications by quasigroup elements, as described in Section 1.3. These two sets are related by the concept of *parastrophy*, extended to arbitrary triply stochastic cubic tensors in Section 3.1. Parastrophy becomes much more transparent within the linear algebraic context: simply transposition of the external indices of the tensor (Definition 3.1). Under parastrophy, strong approximate quasigroups are related to *strong approximate Latin squares* (Section 3.5). In turn, weak approximate Latin squares are related to *weak approximate quasigroups* (Section 2).

1.5. The polytope of triply stochastic cubic tensors

For a given positive degree n, the set Ω_n of all doubly stochastic square matrices forms a polytope of dimension $(n-1)^2$. The permutation matrices are its extreme points [4,5], [15, §II.1.7], [18, Th. II]. This polytope is often described as the *Birkhoff polytope*, in recognition of Birkhoff's initial identification of the extreme points [2, §1]. In turn, the set Λ_n of triply stochastic cubic tensors of degree n also forms a polytope, of dimension $(n-1)^3$ (Section 3.3). However, although each Latin square is an extreme point (Theorem 3.13), in degrees $n > 2$ there are other extreme points which we describe here as *exotic* [11]. Problem 3.21 asks whether strong approximate Latin squares may appear as exotic extreme points of Λ_n. The problem is shown to have a negative answer for $n = 3$ (Proposition 3.22). The convex hull of the Latin squares is described as the *indigenous* part of Λ_n. Problem 5.2 asks whether each strong approximate quasigroup lies in the indigenous part of the polytope Λ_n.

1.6. Set-valued algebras

Binary multiplications of weak and strong approximate quasigroups take values among probability distributions on the underlying set of the algebra. If one is willing to allow a loss of information, one may also pass from probability distributions to their supports, obtaining algebras whose binary multiplication takes values in the set of subsets of the underlying set of the algebra. Such algebras have been studied quite widely. The final section of the paper reviews relevant terminology, shows how the supports of approximate quasigroups satisfy the *reproduction axiom* (Theorem 6.4), and locates the very well-behaved strong approximate quasigroup from Section 5.2 within the context of set-valued algebras.