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Hamiltonian paths in spanning subgraphs of line graphs ☆

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ABSTRACT

The line graph is a very popular research object in graph theory, in complex networks and also in social networks recently. In this paper, we show that if a line graph is Hamiltonian-connected, then the graphs in a special family of spanning subgraphs of the line graph are still Hamiltonian-connected. As an important corollary we prove that there exist at least $max\{1, \lfloor \frac{1}{8}\delta(G) \rfloor - 1\}$ edge-disjoint Hamiltonian paths between any two vertices in a Hamiltonian-connected line graph $L(G)$.

1. Introduction and preliminary

In this paper, a *graph* will be a finite undirected graph $G = (V(G), E(G))$ without loops or multiple edges. For any vertex v of G, we denote the set of edges which are incident with v by $E_G(v)$ and the degree of v in G by $d_G(v)$. A *Hamiltonian path (resp. cycle)* is a spanning path (resp. cycle), i.e., a path (resp. cycle) visits each vertex of the graph exactly once. A graph is *Hamiltonian-connected* if for every pair of vertices there is a Hamiltonian path between the two vertices. A graph is *Hamiltonian* if it contains a Hamiltonian cycle. A Hamiltonian-connected graph is also Hamiltonian, but a Hamiltonian graph may not be Hamiltonian-connected. For terminology and notation not defined here we refer to [3].

The *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$, where two vertices of $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges in G are incident with a common vertex in G. G is called the *original graph* of $L(G)$.

The Hamiltonian path problem (resp. the Hamiltonian cycle problem) is the problem of determining whether a Hamiltonian path (resp. a Hamiltonian cycle) exists in a given graph. It is well-known that these problems are *NP*-complete. The Hamiltonian problem of line graphs has been widely studied by various researchers since Harary and Nash-Williams [4] obtained the first result in this area in 1965. In 1986, Thomassen [11] conjectured that every 4-connected line graph is Hamiltonian. This conjecture is still open now. There are plenty of works on this conjecture. The best result till now is proved by Kaiser and Vrána [5]. They showed that every 5-connected line graph with minimum degree at least 6 is Hamiltonian. We do not aim to survey results related to Thomassen conjecture, see [5,6,8,10] for some nice results.

The Hamiltonian path completion problem is to find the minimal number of new edges to be added to a graph to result in a graph containing a Hamiltonian path. The Hamiltonian path completion problem in line graphs is *NP*-hard (see [2]). Since the line graph $L(G)$ of a graph G has a Hamiltonian path if and only if G has a dominating trail, the Hamiltonian path completion problem in the line graph $L(G)$ is strongly related to the problem of finding a minimum dominating close trail in G.

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On the basis of these problems, in this paper we consider an "inverse" problem of the Hamiltonian path completion problem in line graphs: at most how many edges can be deleted in a Hamiltonian-connected line graph such that the resulted graph is still Hamiltonian-connected? We will give several answers to this problem.

Finding edge-disjoint Hamiltonian paths or cycles is a quite interesting problem to generalize the Hamiltonian problems. The hunt for more than one Hamiltonian path or cycle has a long history. An early result due to Nash-Williams [9] proved that the minimum degree condition (more than half of the graph order) guarantees the existence of many edge-disjoint Hamiltonian cycles (at least $\lfloor \frac{5n}{224} \rfloor$ cycles, n is the graph order). Nash-Williams asked if this number could be improved and this has been a question of interest ever since. For line graphs, Alspach et al. [1] proposed a conjecture about the relations between the edge-disjoint Hamiltonian cycles in line graphs and the edge-disjoint Hamiltonian cycles in the original graphs. Recently, Li et al. [7] prove that there are at least $max\{1, \lfloor \frac{1}{8}\delta(G) - \frac{3}{4} \rfloor\}$ edge-disjoint Hamiltonian cycles in a Hamiltonian line graph $L(G)$. In this paper we consider the edge-disjoint Hamiltonian paths in line graphs and manage to find edge-disjoint Hamiltonian paths between any two vertices in a Hamiltonian-connected line graph based on these results.

The paper is organized as follows. In Section 2, we introduce a family of spanning subgraphs of line graphs and prove that if a line graph is Hamiltonian-connected, then these spanning subgraphs of this line graph are also Hamiltonian-connected. This partially generalizes a result in [7]. In Section 3, we obtain a lower bound on the number of edge-disjoint Hamiltonian paths between any two vertices in a Hamiltonian-connected line graph. A summary and the conclusions addressing also further research directions form the arguments of the last section.

2. Hamiltonian-connectedness in a special family of spanning subgraphs of line graphs

In this section, we will show that for a Hamiltonian-connected line graph $L(G)$, there exist some edge subset X such that $L(G) - X$ is also Hamiltonian-connected.

By the definition of line graphs, for every vertex v in G, the vertices that correspond to all edges in $E_G(v)$ form a clique, for convenience which is also denoted by $E_G(v)$ in the line graph $L(G)$ of G. Now we define a special family of spanning subgraphs of line graphs.

Definition. Let $G = (V(G), E(G))$ be a graph. A subgraph H of $L(G)$ is called $SL(G)$ if

(1) $V(H) = V(L(G))$, and
(2) for every vertex e in H, where $e = uv$ for some vertices $u, v \in V(G)$, e is adjacent in H to at least $min\{d_G(u) - 1, \lceil \frac{3}{4}d_G(u) \rceil + 1\}$ vertices of $E_G(u)$ and also adjacent to at least $min\{d_G(v) - 1, \lceil \frac{3}{4}d_G(v) \rceil + 1\}$ vertices of $E_G(v)$.

Denote the family of all $SL(G)$ by $\mathcal{SL}(G)$. All $SL(G)$ can be obtained from $L(G)$ by deleting edges and $L(G) \in \mathcal{SL}(G)$. For any positive integer d, by simple calculation, we have the equality $d = \lceil \frac{3}{4}d \rceil + \lfloor \frac{1}{4}d \rfloor$ and hence for any vertex $v \in V(G)$, $d_G(v) - 1 - (\lceil \frac{3}{4}d_G(v) \rceil + 1) = \lfloor \frac{1}{4}d_G(v) \rfloor - 2$. From $L(G)$ for any vertex $e = uv$, if no more than $max\{0, \lfloor \frac{1}{4}d_G(v) \rfloor - 2\}$ edges incident to e in the clique $E_G(v)$ and no more than $max\{0, \lfloor \frac{1}{4}d_G(u) \rfloor - 2\}$ edges incident to e in the clique $E_G(u)$ are deleted, then the resulting spanning subgraph of $L(G)$ is in $\mathcal{SL}(G)$.

Our main result in this section is the following one.

Theorem 1. *Given a graph G, if $L(G)$ is Hamiltonian-connected, then every $SL(G) \in \mathcal{SL}(G)$ is also Hamiltonian-connected.*

In [7], the authors proved the following result. But we need to mention that the definition of $\mathcal{SL}(G)$ in [7] is slightly different from the definition in this paper. Since a Hamiltonian-connected graph is Hamiltonian, Theorem 1 partially generalizes the following result in [7].

Theorem 2 ([7]). *Given a graph G, if $L(G)$ is Hamiltonian, then every $SL(G) \in \mathcal{SL}(G)$ is also Hamiltonian.*

Before we prove Theorem 1, we need more notation. If $SL(G) = L(G)$, Theorem 1 is meaningless. So we assume that $SL(G)$ has less edges than $L(G)$. These missed edges of $L(G)$ are called *fake edges* in $L(G)$. We call the edges in $SL(G)$ the *non-fake edges* in $L(G)$. Actually, fake edges are those edges which belong to $L(G)$, but do not belong to $SL(G)$. For convenience, we denote the edge of $L(G)$ or $SL(G)$ by the form of ef , if e and f are two edges incident with a common vertex in G. Suppose that P is a Hamiltonian path of $L(G)$. We assign an orientation to P and for any $e \in V(P)$, denote by e^- the vertex preceding e on P and e^+ the vertex following e. Naturally e^{++} is the vertex following e^+ on P and e^{--} is the vertex preceding e^- on P. For e, f on P, let $P(e, f)$ be the subpath of P from e^+ to f^- and let $\bar{P}(f, e)$ be the subpath of P from f^- to e^+. For any $e \in V(P)$ (but not the end vertex of P), if e^-, *e and* e^+ are all in the same set $E_G(v)$ (v is any vertex of G), e is called a *stable* vertex of P; if else, e is called an *unstable* vertex of P. For convenience, we say this property is the *stability* of e. If in some other new Hamiltonian path P_1, the stability of e is not changed, i.e. stable in both P and P_1 or unstable in both P and P_1, we say that e keeps the stability.

Now we begin our proof of Theorem 1.

Proof. We prove Theorem 1 by contradiction. To the contrary, we assume that there exists an $SL(G)$ which is not Hamiltonian-connected, without loss of generality, suppose that for two vertices $x, y \in V(SL(G))$ there is no Hamiltonian path between them in $SL(G)$. We assume that P is a Hamiltonian path between x and y in $L(G)$ such that among all of the Hamiltonian paths joining x and y in $L(G)$,

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