Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Directed strongly regular graphs with rank 6

Junye Ma [a], Bicheng Zhang [a,*], Jinwang Liu [b], Jutao Zhao [c]

[a] Department of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan, 411105, PR China
[b] Department of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, Hunan, 411201, PR China
[c] Department of Mathematics, Changzhi University, Changzhi, Shanxi, 046001, PR China

A R T I C L E I N F O

Article history:
Received 22 September 2015
Received in revised form 11 April 2016
Accepted 24 October 2016
Available online xxxx

Keywords:
Directed strongly regular graphs
Adjacency matrix
Rank of matrix

A B S T R A C T

In this paper, we study 24 possibilities of directed strongly regular graphs with adjacency matrix of rank 6, our proof is based on two effective algorithms. At last, we get 4 families of directed strongly regular graphs with realizable parameter sets, exclude 20 families of directed strongly regular graphs with feasible parameter sets; partially solve the classification problem of directed strongly regular graphs with adjacency matrix of rank 6.

1. Introduction

A directed strongly regular graph (DSRG) with parameters (n, k, μ, λ, t) is a k-regular directed graph on n vertices such that every vertex is on t 2-cycles (which may be thought of as undirected edges), and the number of paths of length two from a vertex x to a vertex y is λ if there is an edge directed from x to y and it is μ otherwise. A DSRG with $t = k$ is an (undirected) strongly regular graph. Duval [1] showed that DSRGs with $t = 0$ are the doubly regular tournaments. It is therefore usually assumed that $0 < t < k$.

Jørgensen [2] gave a characterization of DSRGs whose adjacency matrix has rank 3 or 4, proved that there exists a DSRG with parameters (n, k, μ, λ, t) and with adjacency matrix of rank 3 if and only if the parameters are either $(6m, 2m, m, 0, m)$ or $(8m, 4m, 3m, m, 3m)$, for some positive integer m, and there exists a DSRG with parameters (n, k, μ, λ, t) and with adjacency matrix of rank 4 if and only if the parameters are either $(6m, 3m, 2m, m, 2m)$ or $(12m, 3m, m, 0, m)$, for some positive integer m.

Jørgensen [3] studied the relation between the number of vertices of a DSRG n, regular of degree k and rank of adjacency matrix r, determined the set B_r of all values of $\frac{k}{n}$ for which there exists a k-regular $n \times n$ matrix with rank r.

Jørgensen [4] gave a characterization of DSRGs whose adjacency matrix has rank 5, proved that there exists a DSRG with parameters (n, k, μ, λ, t) and with adjacency matrix of rank 5 if and only if the parameters are either $(20m, 4m, m, 0, m)$, $(36m, 12m, 5m, 2m, 5m)$, $(10m, 4m, 2m, m, 2m)$, $(16m, 8m, 5m, 3m, 5m)$, $(20m, 12m, 9m, 6m, 9m)$ or $(18m, 12m, 10m, 7m, 10m)$, for some positive integer m.

In this paper, we study DSRGs with adjacency matrix of rank 6. From [3] we have: $B_6 = \{\frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{3}{11}, \frac{2}{7}, \frac{3}{10}, \frac{4}{13}, \frac{1}{3}, \frac{5}{14}, \frac{4}{11}, \frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{5}{12}, \frac{3}{7}, \frac{7}{16}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \frac{7}{15}, \frac{8}{17}, \frac{1}{2}, \frac{9}{17}, \frac{8}{15}, \frac{7}{13}, \frac{6}{11}, \frac{5}{9}, \frac{9}{16}, \frac{4}{7}, \frac{7}{12}, \frac{3}{5}, \frac{8}{13}, \frac{5}{8}, \frac{7}{11}, \frac{9}{14}, \frac{2}{3}, \frac{9}{13}, \frac{7}{10}, \frac{5}{7}, \frac{8}{11}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}\}$, where B_6 is the set of all values of $\frac{k}{n}$ for which there exists a k-regular $n \times n$ matrix with rank 6. Let A be a k-regular $n \times n$ matrix with rank 6 and $\frac{k}{n} = q$, where $q \in \{\frac{1}{6}, \frac{2}{9}, \frac{3}{11}, \frac{3}{10}, \frac{4}{13}, \frac{1}{3}, \frac{5}{14}, \frac{5}{13}, \frac{7}{16}, \frac{8}{17}, \frac{1}{2}, \frac{9}{17}, \frac{9}{16}, \frac{7}{12}, \frac{8}{13}, \frac{7}{11}, \frac{9}{14}, \frac{2}{3}, \frac{9}{13}, \frac{7}{10}, \frac{8}{11}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}\}$. We only consider the 24 values of $\frac{k}{n}$.

* Corresponding author.
E-mail address: zhangbicheng@126.com (B. Zhang).

http://dx.doi.org/10.1016/j.disc.2016.10.013
0012-365X/

2. DSRGs with realizable parameters and with rank 6

Lemma 1 (See [1]). *If G is a DSRG with (n, k, μ, λ, t) and with adjacency matrix A, then the complementary graph $\overline{G}$ is a DSRG with $(n, \bar{k}, \bar{\mu}, \bar{\lambda}, \bar{t})$, where $\bar{k} = n - k - 1$, $\bar{\lambda}$=$(n - 2k) + (\mu - 2)$, $\bar{t} = (n - 2k) + (t + 1)$, $\bar{\mu}$=$(n - 2k) + \lambda$.*

Lemma 2 (See [4]). *Suppose that there exists a DSRG with parameters (n, k, μ, λ, t), then the parameters satisfy*

$$k(k + (\mu - \lambda)) = t + (n - 1)\,\mu, \tag{1}$$

and

$$0 \le \lambda < t, 0 < \mu \le t, -2\,(k - t - 1) \le \mu - \lambda \le 2\,(k - t)\,, \tag{2}$$

the eigenvalues of the adjacency matrix are

$$k > \rho = \frac{1}{2}\,(-\,(\mu - \lambda) + d) > \sigma = \frac{1}{2}\,(-\,(\mu - \lambda) - d)\,,$$

for some positive integer d, where $d^2 = (\mu - \lambda)^2 + 4\,(t - \mu)$. The multiplicities are

$$1, -\frac{k + \sigma\,(n - 1)}{\rho - \sigma}, \frac{k + \rho\,(n - 1)}{\rho - \sigma}, \tag{3}$$

respectively.

We say that (n, k, μ, λ, t) is a feasible parameter set if the conditions (1) and (2) are satisfied and the multiplicities in (3) are positive integers, and we say that (n, k, μ, λ, t) is a realizable parameter set if there exists a DSRG with these parameters.

Lemma 3 (See [1]). *Let A be the adjacency matrix of a DSRG with parameters (n, k, μ, λ, t) $(A \ne J_n - I_n)$, and J_m be the $m\times m$ matrix of all 1's $(m > 1)$. Then $A\times J_m$ is the adjacency matrix of a DSRG if and only if $t = \mu$. In this case the parameter set for $A\times J_m$ is $(nm, km, \mu m, \lambda m, tm)$. The result also holds for $J_m\times A$.*

Lemma 4 (See [4]). *If (n, k, μ, λ, t) are the parameters of a DSRG with $t = \mu$ and rank r and with $\frac{k}{n} = \frac{a}{b}$, where a and b are relatively prime integers, then*

$$(n, k, \mu, \lambda, t) = \left(\frac{(r-1)b^2}{c}m, \frac{(r-1)ab}{c}m, \frac{ra^2}{c}m, \frac{(ar-b)a}{c}m, \frac{ra^2}{c}m\right),$$

for some positive integer m, where c is the greatest common divisor

$$c = \gcd(ab, ra^2, (r - 1)b^2).$$

DSRGs with parameters $(30, 5, 1, 0, 1)$ and $(15, 4, 1, 1, 2)$ were constructed by Duval in [1]. The complementary graph of a DSRG with parameters $(15, 4, 1, 1, 2)$ has parameters $(15, 10, 8, 6, 8)$. Jørgensen constructed a DSRG with parameters $(15, 5, 2, 1, 2)$. In [5], Olmez constructed DSRGs with parameters $(n, k, \mu, \lambda, t) = (ls^2+s, ls+s-1, l+1, l+s-2, l+s-1)$, where l, s are positive integers. This proves that a DSRG with parameters $(10, 5, 3, 2, 3)$ exists. From Lemmas 3 and 4, we can get easily Theorem 5.

Theorem 5. *There exists a DSRG with parameters (n, k, μ, λ, t) and with adjacency matrix of rank 6 if only the parameters are either $(30m, 5m, m, 0, m)$, $(15m, 10m, 8m, 6m, 8m)$, $(15m, 5m, 2m, m, 2m)$ or $(10m, 5m, 3m, 2m, 3m)$, for some positive integer m.*

3. DSRGs with unrealizable parameters and with rank 6

In this section, we consider $q = \frac{2}{9}, \frac{3}{11}, \frac{3}{10}, \frac{4}{13}, \frac{5}{14}, \frac{5}{13}, \frac{7}{16}, \frac{8}{17}, \frac{9}{17}, \frac{9}{16}, \frac{7}{12}, \frac{8}{13}, \frac{7}{11}, \frac{9}{14}, \frac{9}{13}, \frac{7}{10}, \frac{8}{11}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}$. Let A be a k-regular $n\times n$ $(0, 1)$-matrix with rank 6 and $\frac{k}{n} = q$, then $J - A$ is an $(n - k)$-regular $n\times n$ $(0, 1)$-matrix of rank 6 with ratio $\frac{n-k}{n} = 1 - q$.

Lemma 6. *Let A be a k-regular $n\times n$ $(0, 1)$-matrix with m identical rows (columns), then $J-A$ also has m identical rows (columns).*

We can easily get Lemma 6. Thus A and $J - A$ have the same properties with respect to identical rows and identical columns, we only consider $q < \frac{1}{2}$ in the following. We let $j_n \in R^n$ denote the vector with all entries equal to 1.

Theorem 7. *Let A be a k-regular $n\times n$ $(0, 1)$-matrix with rank r and $\frac{k}{n} = q$. Let M be a $r\times n$ submatrix of A of rank r. Let C be a $r\times r$ submatrix of M of rank r with rows $r_1, \ldots, r_r$, columns $c_1, \ldots, c_r$. Let N be a $n\times r$ submatrix of A which regards C as its submatrix. Then the following conditions are all satisfied:*

(1) *there exist unique real numbers $\alpha_1, \ldots, \alpha_r$ so that $\sum_{i=1}^{r}\alpha_i r_i = j_r^T$, $\alpha_1 + \cdots + \alpha_r = \frac{1}{q}$;*
(2) *any row of N can be written as $\sum_{i=1}^{r}\kappa_i r_i$, where $\kappa_1 + \cdots + \kappa_r = 1$;*