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Trivalent vertex-transitive bi-dihedrants

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ABSTRACT

A graph is said to be a *bi-Cayley graph* over a group H if it admits H as a semiregular automorphism group with two vertex-orbits. A *bi-dihedrant* is a bi-Cayley graph over a dihedral group. In this paper, it is shown that every connected trivalent edge-transitive bi-dihedrant is also vertex-transitive, and then we present a classification of trivalent arc-transitive bi-dihedrants, and study the Cayley property of trivalent vertex-transitive bi-dihedrants.

1. Introduction

All groups considered in this paper are finite, and all graphs are finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [5,30].

Given a group H, let R, L and S be subsets of H such that $R^{-1} = R$, $L^{-1} = L$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* over H relative to the triple (R, L, S), denoted by $\mathrm{BiCay}(H, R, L, S)$, is the graph having vertex set the union of the right part $H_0 = \{h_0 \mid h \in H\}$ and the left part $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the right edges $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the left edges $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the spokes $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. For the case when $|S| = 1$, $\mathrm{BiCay}(H, R, L, S)$ is also called a *one-matching bi-Cayley graph*. If $|R| = |L| = s$, then $\mathrm{BiCay}(H, R, L, S)$ is said to be an *s-type bi-Cayley graph*. Note that a graph is isomorphic to a bi-Cayley graph over a group H if and only if it admits a group isomorphic to H as a semiregular group of automorphisms with two vertex-orbits.

Bi-Cayley graphs form an extensively studied class of graphs (see [1,2,9,13,19,21–23]). As one of the most important finite graphs, the Petersen graph is a bi-Cayley graph over a cyclic group of order 5. A bi-Cayley graph over a cyclic group is sometimes simply called a *bicirculant*. The Petersen graph is the initial member of a family of graphs $P(n, t)$, known now as the *generalized Petersen graphs* (see [29]), which can be also constructed as bi-circulants. Let $n \geq 3$, $1 \leq t < n/2$ and set $H = \langle a \rangle \cong \mathbb{Z}_n$. The generalized Petersen graph $P(n, t)$ is isomorphic to the bicirculant $\mathrm{BiCay}(H, \{a, a^{-1}\}, \{a^t, a^{-t}\}, \{1\})$. The complete classification of vertex-transitive (edge-transitive) generalized Petersen graphs has been worked out in [12,25].

Recently, in the study of bi-Cayley graphs, considerable attention was given to the following natural problem: for a given finite group H, classify bi-Cayley graphs with specific symmetry properties over H. For example, all trivalent vertex-transitive (edge-transitive) bicirculants were classified in [24,26], all tetravalent edge-transitive bicirculants were characterized in [16], a classification of arc-transitive one-matching abelian bi-Cayley graphs (namely, bi-Cayley graphs over abelian groups) was presented in [17], and all trivalent vertex-transitive abelian bi-Cayley graphs were classified in [34]. The object of this paper is to characterize trivalent vertex-transitive bi-Cayley graphs over dihedral groups. A bi-Cayley graph over a dihedral group is also simply called a *bi-dihedrant*. Before stating the main results, we introduce some terminology.

For a bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$ over a group H, we can assume that the identity 1 of H is in S (see Proposition 3.1(2)). The triple (R, L, S) of three subsets R, L, S of a group H is called *bi-Cayley triple* if $R = R^{-1}$, $L = L^{-1}$, and

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Table 1
Trivalent edge-transitive bi-dihedrants.

No.	n	$(R, L, S) \equiv$	Γ	Conditions	Cayley
1	$2m$	$(\{b\}, \{ba^{2t}\}, \{1, a\})$	CQ(t, m)	$2 \le t \le m-3$, $m \mid t^2+t+1$	Yes
2	4	$(\{b,\ ba\}, \{ba^2,\ ba^3\}, \{1\})$	F016A		Yes
3	4	$(\{b\}, \{ba\}, \{1, a\})$	F016A		Yes
4	5	$(\{b,\ ba^3\}, \{ba,\ ba^2\}, \{1\})$	F020B		No
5	5	$(\{b, ba\}, \{a, a^{-1}\}, \{1\})$	F020A		No
6	6	$(\{b,\ ba\}, \{ba^3,\ ba^4\}, \{1\})$	F024A		Yes
7	6	$(\{b\}, \{ba^2\}, \{1, a\})$	F024A		Yes
8	8	$(\{b,\ ba\}, \{ba^2,\ ba^5\}, \{1\})$	F032A		Yes
9	10	$(\{b,\ ba^4\}, \{ba,\ ba^3\}, \{1\})$	F040A		No
10	10	$(\{b, ba^4\}, \{a, a^{-1}\}, \{1\})$	F040A		No
11	12	$(\{b,\ ba\}, \{ba^3,\ ba^{10}\}, \{1\})$	F048A		Yes
12	20	$(\{b,\ ba^{14}\}, \{ba,\ ba^3\}, \{1\})$	F080A		No
13	$2m$	$(\{b, ba\}, \{ba^{-2t}, ba^{-2t-1}\}, \{1\})$	CQ(t, m)	$2 \le t \le m-3$ $m \mid t^2-t+1$	Yes
14	$2m$	$(\{b, ba\}, \{ba^{-2t}, ba^{-2t+m-1}\}, \{1\})$,	CQ(t, m)	$2 \le t \le m-3$ $m \mid 2(t^2-t+1)$, m even, t odd	Yes

$1 \in S$. Two bi-Cayley triples (R, L, S) and (R', L', S') of a group H are said to be *equivalent*, denoted by $(R, L, S) \equiv (R', L', S')$, if either $(R', L', S') = (R, L, S)^\alpha$ or $(R', L', S') = (L, R, S^{-1})^\alpha$ for some automorphism α of H. The bi-Cayley graphs corresponding to two equivalent bi-Cayley triples of the same group are isomorphic (see Proposition 3.1(3)–(4)). Hereafter, the notation FnA, FnB, etc. will refer to the corresponding graphs in the Foster census [7,28]. The notation CQ(t, m) will refer to the cyclic cover of the cube F008A (see Section 4 for its construction).

Our first result classifies all trivalent edge-transitive bi-dihedrants.

Theorem 1.1. *Let $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle (n \ge 3)$. A connected trivalent bi-dihedrant $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$ is edge-transitive if and only if the triple (R, L, S) is equivalent to one of the triples in Table 1. Furthermore, all of the graphs in Table 1 are arc-transitive.*

Our second result shows that every trivalent vertex-transitive 0- or 1-type bi-dihedrant is a Cayley graph.

Theorem 1.2. *Every connected trivalent vertex-transitive 0- or 1-type bi-dihedrant is a Cayley graph.*

For the 2-type case, the situation becomes a bit more complicated. We shall first prove the following Theorem 1.3, and using this, in Corollary 1.4 we give a classification of connected trivalent 2-type vertex-transitive non-Cayley bi-Cayley graphs over D_{2n} with n odd. For the case when n is even, the method used here is invalid, and this case shall be dealt with in our subsequent paper [33].

Before stating Theorem 1.3, we first introduce some notation. Let $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$ be a bi-Cayley graph over a group H. It is easy to see that H acts as a semiregular group of automorphisms by right multiplication, with H_0 and H_1 as its orbits on vertices, and we denote this group of automorphisms of Γ as $\mathcal{R}(H)$.

Theorem 1.3. *Let $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$ be a connected trivalent vertex-transitive 2-type bi-dihedrant, where $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle (n \ge 3)$. Suppose that G is a vertex-transitive group of automorphisms of Γ such that $\mathcal{R}(H) \le G$ and H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$. Then, either Γ is Cayley or one of the following occurs:*

(1) *$(R, L, S) \equiv (\{b,\ ba^{\ell+1}\}, \{ba,\ ba^{\ell^2+\ell+1}\}, \{1\})$, where $n \ge 5$, $\ell^3+\ell^2+\ell+1 \equiv 0 \pmod n$, $\ell^2 \not\equiv 1 \pmod n$;*
(2) *$(R, L, S) \equiv (\{ba^{-\ell},\ ba^{\ell}\}, \{a,\ a^{-1}\}, \{1\})$, where $n = 2k$ and $\ell^2 \equiv -1 \pmod k$. Furthermore, Γ is also a bi-Cayley graph over an abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$.*

Furthermore, all of the graphs arising from (1)-(2) are vertex-transitive non-Cayley.

Our last result classifies connected trivalent 2-type vertex-transitive non-Cayley bi-Cayley graphs over D_{2n} with n odd[1].

Corollary 1.4. *Let $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ \{1\})$ be a connected trivalent 2-type vertex-transitive bi-dihedrant, where $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ with n odd. Then Γ is non-Cayley if and only if one of the following occurs:*

[1] By checking the census of trivalent vertex-transitive graphs of order up to 1000 in [27], we find out that there are 981 non-Cayley graphs, and among these graphs, 233 graphs are non-Cayley bi-dihedrants and 187 graphs are covered by Theorem 1.3. This may suggest bi-dihedrants form an important classes of trivalent vertex-transitive non-Cayley graphs. We thank a referee for suggesting us to check the census in [27].