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Planar graphs without intersecting 5-cycles are 4-choosable

Dai-Qiang Hu [a], Jian-Liang Wu [b,*]

[a] Department of Mathematics, Jinan University, Guang Zhou, 510632, PR China
[b] School of Mathematics, Shandong University, Jinan, 250100, PR China

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ABSTRACT

A graph G is k-choosable if it can be properly colored whenever every vertex has a list of at least k available colors. In the paper, it is proved that all planar graphs without intersecting 5-cycles are 4-choosable.

1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for the terminologies and notations not defined here. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. A *k-vertex*, a *k^+-vertex* or a *k^--vertex* is a vertex of degree k , at least k or at most k respectively. We use $\delta(G)$ to denote the minimum degree of G. A k-cycle is a cycle of length k, and a 3-cycle is usually called a *triangle*. Two cycles are *adjacent*(or *intersecting*) if they share at least one edge (or vertex, respectively).

A *proper k-coloring* of a graph G is a mapping ϕ from $V(G)$ to the color set $[k] = \{1, 2, \dots, k\}$ such that $\phi(x) \neq \phi(y)$ for every two adjacent vertices x and y of G. We say that L is an *assignment* for the graph G if it assigns a list $L(v)$ of possible colors to each vertex v of G. If G has a proper coloring ϕ such that $\phi(v) \in L(v)$ for any vertex v, then we say that G is L-colorable or ϕ is an L-coloring of G. The graph G is *k-choosable* if it is L-colorable for every assignment L satisfying $|L(v)| \geq k$ for any vertex v.

The concept of a list coloring was introduced by Vizing [10] and Erdős, Rubin and Taylor [4], respectively. Thomassen [9] showed that every planar graph is 5-choosable. Examples of plane graphs which are not 4-choosable and plane graphs of girth 4 which are not 3-choosable were given by Voigt [11,12]. Since every planar graph G without 3-cycles has a vertex of degree at most 3, it is 4-choosable. Wang and Lih [13] extended the result to all planar graphs without intersecting 3-cycles. Lam, Xu and Liu [8] proved that all planar graphs without 4-cycles are 4-choosable. Lam, Shiu and Xu [7](Wang and Lih [14], respectively) proved that all planar graphs without 5-cycles are 4-choosable. Fijavž, Juvan, Mohar, Škrekovski [6] proved that all planar graphs without 6-cycles are 4-choosable. Farzad [5] proved that all planar graphs without 7-cycles are 4-choosable. Cheng, Chen and Wang [3] proved that planar graphs without 4-cycles adjacent to triangles are 4-choosable. In the paper, we prove that all planar graphs without intersecting 5-cycles are 4-choosable.

* Corresponding author.
E-mail address: jlwu@sdu.edu.cn (J. Wu).

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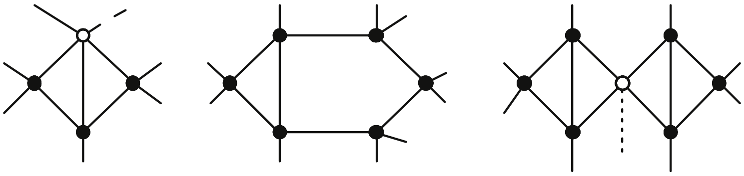

Fig. 1. Three subgraphs of G.

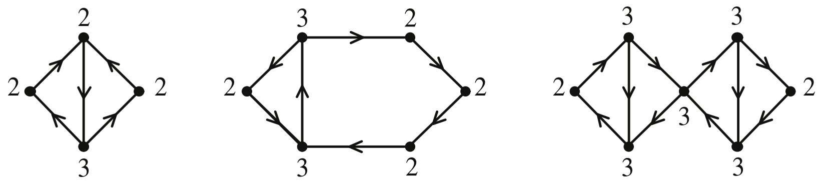

Fig. 2. An orientation of the configurations in Figure 1.

2. Main result and its proof

First, we introduce some notations and definitions used in this section. Let G be a plane graph. We use F or $F(G)$ to denote the face set of G. For $f \in F(G)$, we use $V(f)$ to denote the set of vertices on the boundary of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in a cyclic order. A face of G is said to be *incident* with all edges and vertices in its boundary. The *degree* of a face f, denoted by $d_G(f)$, is the number of edges incident with it, where a cut edge is counted twice. A *k-face*, *k^--face* or a *k^+-face* is a face of degree k, at most k or at least k, respectively. For convenience, a k-face $f = [v_1v_2 \cdots v_k]$ is often said to be a $(d(v_1), d(v_2), \ldots, d(v_k))$-face. For a face f, let $n_i(f)$ and $n_{i^+}(f)$ denote the number of i-vertices and i^+-vertices incident with f, respectively. Denote by $f_d(v)$ and $f_{d^+}(v)$ the number of d-faces and d^+-faces incident with v, respectively.

Theorem 1. *All planar graphs without intersecting* 5*-cycles are* 4*-choosable.*

Proof. Suppose, to the contrary, that Theorem 1 is false. Let G be a counterexample to Theorem 1 with the fewest vertices. Then $\delta(G) \geq 4$ (see [8]).

First, we introduce a well-known theorem proved by Alon and Tarsi [1]. This intricate theorem reveals the connection between the list coloring of a graph G and its orientations. A digraph (directed graph) D is an ordered pair $(V(D), A(D))$ consisting of the vertex set $V(D)$ and arc set $A(D)$. For any arc $a = \langle u, v \rangle$, we say that u is the tail of a and v its head. The indegree $d_D^-(v)$ of a vertex v in D is the number of arcs with head v, and the outdegree $d_D^+(v)$ of v is the number of arcs with tail v. A directed cycle is denoted by a cyclic sequence $u_1u_2 \cdots u_ku_1$ in which each vertex dominates its successor. A subdigraph H of a directed graph D is called *Eulerian* if the indegree $d_H^-(v)$ of every vertex v of H is equal to its outdegree $d_H^+(v)$. Note that we do not assume that H is connected. H is *even* if it has an even number of edges. Otherwise it is *odd*. Let $EE(D)$ and $EO(D)$ denote the numbers of even and odd Eulerian subgraphs of D, respectively. (For convenience we agree that the empty subgraph is an even Eulerian subgraph).

Theorem 2. [1] *Let D be a digraph. For each vertex $v \in V(D)$, let $f(v)$ be a set of $d_D^+(v) + 1$ distinct integers, where $d_D^+(v)$ is the outdegree of v. If $EE(D) \neq EO(D)$, then there is a proper coloring $c : V(D) \to \mathbb{Z}$ such that $c(v)$ is in $f(v)$ for every vertex v. That is, if L is a list assignment such that $|L(v)| = d_D^+(v) + 1$ for all vertices v in D, then D is L-colorable.*

Lemma 3. *G contains no subgraph isomorphic to one of the configurations in Fig. 1, where the vertices marked by $\bullet$ have degree of* 4 *in G and those vertices marked by $\circ$ have degree of* 4 *or* 5 *in G.*

Proof. Let L be an arbitrary list assignment of G such that each vertex is assigned precisely 4 colors. Suppose that G contains a subgraph H isomorphic to one of the configurations in Fig. 1. By the minimality of G, the graph $G - V(H)$ has an L-coloring φ. An orientation of H and available colors of every vertex are shown in Fig. 2. Since $EE(G_1) = 1 < EO(G_1) = 2$, $EE(G_2) = 1 < EO(G_2) = 2$ and $EE(G_3) = 5 > EO(G_3) = 4$, any $G_i (1 \leq i \leq 3)$ satisfies Theorem 2. So φ can be extended to G. ■