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Well-dominated graphs without cycles of lengths 4 and 5

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ABSTRACT

Let G be a graph. A set S of vertices in G *dominates* the graph if every vertex of G is either in S or a neighbor of a vertex in S. Finding a minimum cardinality set which dominates the graph is an **NP**-complete problem. The graph G is *well-dominated* if all its minimal dominating sets are of the same cardinality. The complexity status of recognizing well-dominated graphs is not known. We show that recognizing well-dominated graphs can be done polynomially for graphs without cycles of lengths 4 and 5, by proving that a graph belonging to this family is well-dominated if and only if it is well-covered.

Assume that a weight function w is defined on the vertices of G. Then G is w-*well-dominated* if all its minimal dominating sets are of the same weight. We prove that the set of weight functions w such that G is w-well-dominated is a vector space, and denote that vector space by $WWD(G)$. We show that $WWD(G)$ is a subspace of $WCW(G)$, the vector space of weight functions w such that G is w-well-covered. We provide a polynomial characterization of $WWD(G)$ for the case that G does not contain cycles of lengths 4, 5, and 6.

1. Introduction

1.1. Definitions and notations

Throughout this paper G is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V(G)$ and edge set $E(G)$.

Cycles of k vertices are denoted by C_k. When we say that G does not contain C_k for some $k \geq 3$, we mean that G does not admit subgraphs isomorphic to C_k. It is important to mention that these subgraphs are not necessarily induced. Let $\mathcal{G}(\widehat{C_{i_1}}, \ldots, \widehat{C_{i_k}})$ be the family of all graphs which do not contain $C_{i_1}, \ldots, C_{i_k}$.

Let u and v be two vertices in G. The *distance* between u and v, denoted $d(u, v)$, is the length of a shortest path between u and v, where the length of a path is the number of its edges. If S is a non-empty set of vertices, then the *distance* between u and S, denoted $d(u, S)$, is defined by:

$$d(u, S) = \min\{d(u, s) : s \in S\}.$$

For every i, denote

$$N_i(S) = \{x \in V(G) : d(x, S) = i\}, N_i[S] = \{x \in V(G) : d(x, S) \leq i\}.$$

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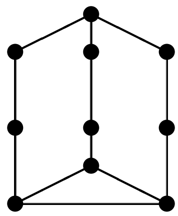

Fig. 1. The graph T_{10}.

We abbreviate $N_1(S)$ and $N_1[S]$ to be $N(S)$ and $N[S]$, respectively. If S contains a single vertex, v, then we abbreviate $N_i(\{v\})$, $N_i[\{v\}]$, $N(\{v\})$, and $N[\{v\}]$ to be $N_i(v)$, $N_i[v]$, $N(v)$, and $N[v]$, respectively.

For every vertex $v \in V(G)$, the *degree* of v is $d(v) = |N(v)|$. Let $L(G)$ be the set of all vertices $v \in V(G)$ such that either $d(v) = 1$ or v is on a triangle and $d(v) = 2$.

1.2. Well-covered graphs

A set of vertices is *independent* if its elements are pairwise nonadjacent. Define $D(v) = N(v) \setminus N(N_2(v))$, and let $M(v)$ be a maximal independent set of $D(v)$. An independent set of vertices is *maximal* if it is not a subset of another independent set. An independent set is *maximum* if G does not admit an independent set with a bigger cardinality. Denote $i(G)$ the minimum cardinality of a maximal independent set in G, and by $\alpha(G)$ the maximum cardinality of an independent set in G.

The graph G is *well-covered* if $i(G) = \alpha(G)$, i.e. all maximal independent sets are of the same cardinality. The problem of finding a maximum cardinality independent set $\alpha(G)$ in an input graph is **NP**-complete. However, if the input is restricted to well-covered graphs, then a maximum cardinality independent set can be found polynomially using the *greedy algorithm*.

Let $w : V(G) \longrightarrow \mathbb{R}$ be a weight function defined on the vertices of G. For every set $S \subseteq V(G)$, define $w(S) = \Sigma_{s \in S} w(s)$. The graph G is w*-well-covered* if all maximal independent sets are of the same weight. The set of weight functions w for which G is w-well-covered is a *vector space* [6]. This vector space is denoted $WCW(G)$ and analyzed in [2–4].

Since recognizing well-covered graphs is **co-NP**-complete [7,16], finding the vector space $WCW(G)$ of an input graph G is **co-NP**-hard. Finding $WCW(G)$ remains **co-NP**-hard when the input is restricted to graphs with girth at least 6 [12], and bipartite graphs [12]. However, the problem is polynomially solvable for $K_{1,3}$-free graphs [13], and for graphs with a bounded maximum degree [12]. For every graph G without cycles of lengths 4, 5, and 6, the vector space $WCW(G)$ is characterized as follows.

Theorem 1 *([14]). Let $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$ be a graph, and let $w : V(G) \longrightarrow \mathbb{R}$. Then G is w-well-covered if and only if one of the following holds:*

1. *G is isomorphic to either C_7 or T_{10} (see Fig. 1), and there exists a constant $k \in \mathbb{R}$ such that $w \equiv k$.*
2. *The following conditions hold:*
   - *G is isomorphic to neither C_7 nor T_{10}.*
   - *For every two vertices, l_1 and l_2, in the same component of $L(G)$ it holds that $w(l_1) = w(l_2)$.*
   - *For every $v \in V(G) \setminus L(G)$ it holds that $w(v) = w(M(v))$ for some maximal independent set $M(v)$ of $D(v)$.*

Recognizing well-covered graphs is a restricted case of finding $WCW(G)$. Therefore, for all families of graphs for which finding $WCW(G)$ is polynomially solvable, recognizing well-covered graphs is polynomially solvable as well. Recognizing well-covered graphs is **co-NP**-complete for $K_{1,4}$-free graphs [5], but it is polynomially solvable for graphs without cycles of lengths 3 and 4 [9], for graphs without cycles of lengths 4 and 5 [10], or for chordal graphs [15].

1.3. Well-dominated graphs

Let S and T be two sets of vertices of the graph G. Then S *dominates* T if $T \subseteq N[S]$. The set S is *dominating* if it dominates all vertices of the graph. A dominating set is *minimal* if it does not contain another dominating set. A dominating set is *minimum* if G does not admit a dominating set with smaller cardinality. Let $\gamma(G)$ be the cardinality of a minimum dominating set in G, and let $\Gamma(G)$ be the maximum cardinality of a minimal dominating set of G. If $\gamma(G) = \Gamma(G)$ then the graph is *well-dominated*, i.e. all minimal dominating sets are of the same cardinality. This concept was introduced in [8], and further studied in [11]. The fact that every maximal independent set is also a minimal dominating set implies that

$$\gamma(G) \le i(G) \le \alpha(G) \le \Gamma(G)$$

for every graph G. Hence, if G is not well-covered, then it is not well-dominated.