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An upper bound on the algebraic connectivity of outerplanar graphs

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ABSTRACT

In this paper, we determine upper bounds on the algebraic connectivity, denoted as $a(\mathcal{G})$, of maximal outerplanar graphs. We show that if $\mathcal{G}$ is a maximal outerplanar graph on $n \geq 12$ vertices not of the form $K_1 \vee P_{n-1}$, then $a(\mathcal{G}) \leq 1$ with equality holding for exactly two maximal outerplanar graphs on 12 vertices. We show this by assigning labels $y_1, \ldots, y_n$ to the vertices and showing the existence of vertex labellings such that $\sum_{uv \in E(\mathcal{G})}(y_u - y_v)^2 / \sum_{v \in V(\mathcal{G})} y_v^2 < 1$.

1. Introduction and preliminaries

An *undirected graph* $\mathcal{G} = (V, E)$ on n vertices is a finite set V of cardinality n, whose elements are called *vertices*, together with a set E of two-element subsets of V called *edges*. It will be convenient to label the vertices $1, 2, \ldots, n$. We say that two vertices i and j are *adjacent* if there is an edge joining i and j. Furthermore, the *degree* of a vertex i, which we denote as $\deg(i)$, is the number of edges incident to i.

With $\mathcal{G}$ we can associate the so called *Laplacian matrix* which is the $n \times n$ matrix $L = (\ell_{i,j})$ whose entries are determined as follows:

$$\ell_{i,j} = \begin{cases} -1, & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ 0, & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ \deg(i), & \text{if } i = j. \end{cases}$$

It is known that the Laplacian matrix is a symmetric positive semidefinite M-matrix.[1] We shall always consider its eigenvalues to have been arranged in a nondescending order: $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. For an extensive survey on the Laplacian matrix see Merris [6].

The eigenvalue $a(\mathcal{G}) := \lambda_2$ of L is known as the *algebraic connectivity* of a graph. Much work has been done concerning the algebraic connectivity of graphs. We refer the reader to [4] and [5] for a survey of such work. For example, Fiedler [3] has shown that $\lambda_2 > 0$ if and only if $\mathcal{G}$ is connected. In this same paper, Fiedler showed that if edges are added to a graph joining two nonadjacent vertices, then the algebraic connectivity does not decrease, and often increases.

A graph is *planar* if it can be drawn on a plane so that none of its edges cross. It is well known (see [10] for instance) that if a graph $\mathcal{G}$ is planar with n vertices and m edges, then $m \leq 3n - 6$. Intuitively, since planar graphs cannot have too many edges, it follows that the algebraic connectivity of planar graphs would be small. It was proven in [8] that if $\mathcal{G}$ is a planar graph, then $a(\mathcal{G}) \leq 4$ and equality holds if and only if $\mathcal{G} \cong K_4$ or $\mathcal{G} \cong K_{2,2,2}$.

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1 For more background material on nonnegative matrices and M-matrices see Berman and Plemmons [2].

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In this paper we investigate the algebraic connectivity of maximal outerplanar graphs. A graph is *outerplanar* if it is planar and if all of its vertices can be placed in a cycle so that all of its edges lie in the interior of the cycle with no edges crossing. In [1], it was shown that if $\mathcal{G}$ is an outerplanar graph on n vertices, then $a(\mathcal{G}) \leq 1 + O(\frac{1}{n})$. In this paper we sharpen this bound by showing that if $\mathcal{G}$ is an outerplanar graph on $n \geq 12$ vertices that is not of the form $K_1 \vee P_{n-1}$, then $a(\mathcal{G}) \leq 1$ where equality holds if and only if $\mathcal{G}$ is one of two specific graphs on 12 vertices. (Note that K_1 is the isolated vertex, $\vee$ denotes the join of two graphs, and P_k denotes the path on k vertices.)

2. Preliminary results

In this section, we recall several results which will be of great use to us. Our first result concerns the algebraic connectivity of $K_1 \vee P_{n-1}$:

Theorem 2.1. $a(K_1 \vee P_{n-1}) = 3 - 2\cos\frac{\pi}{n-1} > 1$.

Proof. It is well-known from [3] that $a(P_k) = 2 - 2\cos\frac{\pi}{k}$. It is also well-known that $a(K_1 \vee \mathcal{G}) = 1 + a(\mathcal{G})$ for any graph $\mathcal{G}$. These two results immediately imply $a(K_1 \vee P_{n-1}) = 3 - 2\cos\frac{\pi}{n-1}$. The inequality follows from the fact that $\cos\frac{\pi}{n-1} < 1$ when $n \geq 2$. $\square$

Therefore, if $\mathcal{G}$ is an outerplanar graph on $n \geq 12$ vertices, we exclude graphs of the from $K_1 \vee P_{n-1}$ when showing that $a(\mathcal{G}) \leq 1$.

Our next result is well known (see [3], for instance) and concerns the effects on the algebraic connectivity when edges are added to a graph.

Theorem 2.2. *Let $\mathcal{G}$ be a non-complete graph and let $\hat{\mathcal{G}}$ be the graph obtained from $\mathcal{G}$ by adding an edge joining two nonadjacent vertices. Then $a(\mathcal{G}) \leq a(\hat{\mathcal{G}})$.*

In light of Theorem 2.2, we need only consider *maximal outerplanar graphs* in our investigation, that is, outerplanar graphs in which adding an edge joining two nonadjacent vertices would render the resulting graph to no longer be outerplanar.

Our next result, also from [3], gives us information about the eigenvector y associated with $a(\mathcal{G})$, better known as the *Fiedler vector*. In this result and in the remainder of this paper, we will let $E(\mathcal{G})$ and $V(\mathcal{G})$ denote the edge set and vertex set of $\mathcal{G}$, respectively. We will let $v_1, \ldots, v_n$ denote the vertices of $\mathcal{G}$. Further, let y_i denote the entry in the Fiedler vector y corresponding to v_i. By a mild abuse of notation, if $v \in V(\mathcal{G})$, we will let y_v, or simply v, denote the entry in the Fiedler vector y corresponding to the vertex v. Finally, let e denote the vector of all $1's$ of the appropriate size. Now for the result:

Theorem 2.3. *Let $\mathcal{G}$ be a graph with Laplacian matrix L, algebraic connectivity $a(\mathcal{G})$, and Fiedler vector y. Then*

$$a(\mathcal{G}) = \min_{y\neq 0, y^T e=0} \frac{\sum_{uv\in E(\mathcal{G})}(y_u - y_v)^2}{\sum_{v\in V(\mathcal{G})} y_v^2}.$$

As a result of Theorem 2.3, we have the following useful corollary:

Corollary 2.4. *Let $\mathcal{G}$ be a graph and let $y_1, \ldots, y_n$ be labellings of the vertices $v_1, \ldots, v_n \in V(\mathcal{G})$ such that $\sum_{i=1}^n y_i = 0$ and where not all y_i are zero. Then*

$$a(\mathcal{G}) \leq \frac{\sum_{uv\in E(\mathcal{G})}(y_u - y_v)^2}{\sum_{v\in V(\mathcal{G})} y_v^2} := w_y(\mathcal{G})$$

where equality holds if and only if the vector $y = [y_1, \ldots, y_n]^T$ is a Fiedler vector for $\mathcal{G}$.

Remark 2.5. Since our goal is to show that $a(\mathcal{G}) \leq 1$ for the outerplanar graphs in question, we will apply Corollary 2.4 by showing that we can label the vertices of such outerplanar graphs in accordance with Corollary 2.4 so that $w_y(\mathcal{G}) < 1$. To simplify our notation, let $D = \sum_{v\in V(\mathcal{G})} y_v^2$ and let $N = \sum_{uv\in E(\mathcal{G})}(y_u - y_v)^2$. (The variables D and N stand for denominator and numerator, respectively.) Hence, our goal will be to prove that it is possible to label the vertices of the outerplanar graphs in question such that the sum of the vertex labellings is zero and $D > N$, or equivalently, $D - N > 0$.

Remark 2.6. It should be noted that this technique, used extensively in [9], is a generalization of the technique used in [7] regarding the isoperimetric number of a graph which is an upper bound on its algebraic connectivity. In [7], the vertex set of a graph $\mathcal{G}$ is partitioned into two sets X and Y where the vertices in X are labelled $|Y|$ and the vertices in Y are labelled $-|X|$, where $|S|$ denotes the cardinality of a set S.

We close this section with some notation that will be useful to us throughout the remainder of this paper. If $\mathcal{G}$ is an outerplanar graph on n vertices, then we will label the vertices $v_1, \ldots, v_n$ so that v_i is adjacent to v_{i+1} for all $i = 1, \ldots, n-1$, and v_n is adjacent to v_1. Thus the vertices $v_1, \ldots, v_n, v_1$ form a cycle which we will refer to as the *outercycle* of an outerplanar