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Classification of nonorientable regular embeddings of cartesian products of graphs

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ABSTRACT

In this paper, we consider the classification of nonorientable regular embeddings of cartesian products G^d of a graph G. We show that if G is a bipartite graph and $d \geq 3$, then there is no nonorientable regular embedding. This is a generalization of the result that there is no nonorientable regular embeddings of Q_n for $n \geq 3$ shown by R. Nedela and the author in 2007. When G is non-bipartite and $d \geq 3$, we show that if there is a nonorientable regular embedding of G^d, then there is a nonorientable regular embedding of G. Furthermore, we show that any nonorientable regular embedding of G^d with $d \geq 3$ is isomorphic to a Petrie dual of some orientable regular embedding of G^d.

Using these results, we classify the nonorientable regular embeddings of cartesian products C_n^d of a cycle C_n; for even n there is no nonorientable regular embedding except when $d = 1$, and for odd n there is a unique nonorientable regular embedding for each d.

1. Introduction

All graphs in this paper are connected graphs except when we refer to subgraphs of the cycle space $\mathcal{C}(G)$. A *map* is a 2-cell embedding of a graph G into a compact, connected surface. An automorphism of a map is an automorphism of a graph G which extends to a self-homeomorphism of the underlying surface. The group $\Gamma := \mathrm{Aut}(\mathcal{M})$ of all automorphisms of $\mathcal{M}$ acts semi-regularly on its *flags* – mutually incident triples of the form vertex–edge–face. If $\mathrm{Aut}(\mathcal{M})$ acts regularly on the flags, then the map $\mathcal{M}$ as well as the corresponding embedding is also called *regular*. Equivalently, $\mathcal{M}$ is regular if and only if there are three involutions λ, ρ and τ in Γ, each fixing a distinct pair of elements v, e, f of some flag (v, e, f) such that $|\langle\lambda, \rho, \tau\rangle| = 4|E(G)|$; in this case we have $\Gamma = \langle\lambda, \rho, \tau\rangle$. In what follows we shall assume that $\mathcal{M}$ is a regular map with λ fixing e and f, and ρ fixing v and f, so τ fixes v and e. We call such a triple (λ, ρ, τ) a *regular embedding triple* of G. If $\mathcal{M}$ is a nonorientable regular map, then (λ, ρ, τ) is called a *nonorientable regular embedding triple* of G. By regularity of the action, $\tau\lambda = \lambda\tau$. Thus the stabilizer $\Gamma_e \cong Z_2 \times Z_2$ is a dihedral group of order 4. Similarly, the stabilizers $\Gamma_v = \langle\rho, \tau\rangle$ and $\Gamma_f = \langle\lambda, \rho\rangle$ of v and f are dihedral groups of orders $2m$ and $2n$, where m and n are the valency and the covalency of $\mathcal{M}$, respectively. So the order of G is $4|E(G)|$. When a regular map $\mathcal{M}$ is represented in this way by a triple of involutions λ, ρ, τ, we write $\mathcal{M} = \mathcal{M}(\lambda, \rho, \tau)$.

Conversely, for a graph G and for three involutions $\lambda, \rho, \tau \in \mathrm{Aut}(G)$ such that (i) ρ, τ fix a vertex v and the cyclic subgroup $\langle\rho\tau\rangle$ acts regularly on the arcs emanating from v and (ii) λ, τ fix an edge $e = \{u, v\}$ incident to v, $v^\lambda = u$ and $\lambda\tau = \tau\lambda$, the triple (λ, ρ, τ) is a regular embedding triple of G if and only if the order of the group $\langle\lambda, \rho, \tau\rangle$ is $4|E(G)|$. Furthermore, for a regular embedding triple (λ, ρ, τ) of G, there is a corresponding regular embedding $\mathcal{M}$ of G such that $\mathrm{Aut}(\mathcal{M}) = \langle\lambda, \rho, \tau\rangle$, namely $\mathcal{M} = \mathcal{M}(\lambda, \rho, \tau)$.

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Two regular maps $\mathcal{M}(\lambda, \rho, \tau)$ and $\mathcal{M}'(\lambda', \rho', \tau')$ with the underlying graphs G and G' are *isomorphic* if there is a graph isomorphism ψ from G to G' such that $\psi^{-1}\lambda\psi = \lambda'$, $\psi^{-1}\rho\psi = \rho'$ and $\psi^{-1}\tau\psi = \tau'$. A detailed explanation of the above representation of regular maps can be found in [1, Theorem 3]. The basics of the theory of regular maps as well as some more information can be found in papers [7,8,12–14].

If a regular map is determined by an embedding $i : G \mapsto S$ of a graph G into a surface S, we say that i is a *regular embedding* of G. The surface underlying a map $\mathcal{M}$ is nonorientable if and only if there is a cycle C whose regular neighbourhood forms a subspace homoeomorphic to a Möbius band. Such a cycle will be called *reversible*. If $\mathcal{M}$ is regular with $\mathrm{Aut}\,\mathcal{M} = \langle \lambda, \rho, \tau \rangle$ then the nonorientability of S is reflected by the fact that the even word subgroup $\mathrm{Aut}^+(\mathcal{M}) = \langle \rho\tau, \tau\lambda \rangle = \langle R, L \rangle$ is the full automorphism group $\mathrm{Aut}\,(\mathcal{M})$, where $R = \rho\tau$ and $L = \tau\lambda$. In particular, if there exists a reversible cycle C of length n in $\mathcal{M}$ then there is an associated identity of the form $LR^{m_1}LR^{m_2}\dots LR^{m_n} = \tau$ in $\mathrm{Aut}\,(\mathcal{M})$. Vice-versa, an existence of such an identity in $\mathrm{Aut}\,(\mathcal{M})$ forces $\mathrm{Aut}\,(\mathcal{M}^+) = \mathrm{Aut}\,(\mathcal{M})$, so $\mathcal{M}$ is nonorientable.

Only for few families of graphs a complete classification of their regular embeddings is known. Such embeddings of complete graphs K_n have been classified by James [5] and Wilson [15]: these exist if and only if n is 3, 4 or 6. Nedela and the author [10] have shown the nonexistence of a nonorientable regular embedding of the n-dimensional cube graph Q_n for all n except $n = 2$. Jones and the author [6] have shown that there exists a regular embedding of the Hamming graph $H(d, n)$ if and only if $n = 2$ and $d = 2$, or $n = 3$ or 4 and $d \geq 1$, or $n = 6$ and $d = 1$ or 2. In contrast with all other known cases, the complete bipartite graph $K_{n,n}$ has a nonorientable regular embedding for infinitely many values of n, as shown by Kwak and the author [9]: in fact, such an embedding exists if and only if $n \equiv 2 \pmod 4$ and all odd prime divisors of n are congruent to $\pm 1 \pmod 8$.

Our paper is organized as follows. In the next section, several properties of cartesian products of graphs will be considered. In Section 3, we deal with nonorientable regular embeddings of cartesian products of graphs. In Sections 4 and 5, we classify nonorientable regular embeddings of cartesian products C_n^d of cycles C_n.

2. Cartesian products of graphs

For graphs G and H, the *cartesian product* of G and H denoted by $G\Box H$ is the graph with the vertex set $V(G) \times V(H)$ such that for two vertices $(g_1, h_1), (g_2, h_2) \in V(G)\times V(H)$, (g_1, h_1) and (g_2, h_2) are adjacent in $G\Box H$ if and only if either $h_1 = h_2$ and $g_1g_2 \in E(G)$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$. In $G\Box H$, the graphs G and H are called *factors* of $G\Box H$. For two edges $e_1 = g_1g_2 \in E(G)$, $e_2 = h_1h_2 \in E(H)$, the product $e_1\Box e_2$ is a 4-cycle in $G\Box H$. We call such 4-cycles *parallel 4-cycles* of $G\Box H$. A graph G is called a *prime graph* if G cannot be described by a cartesian product of two nontrivial graphs. Note that the cartesian product operation is commutative and associative up to isomorphism. For given graphs $G_0, \dots, G_{d-1}$, the cartesian product $G_0\Box\cdots\Box G_{d-1}$ is denoted by $\prod_{i=0}^{d-1} G_i$. Especially if $G_0 = \cdots = G_{d-1}$, then $\prod_{i=0}^{d-1} G_i$ is simply denoted by G^d. For any $j = 0, \dots, d-1$ and for any $v_i \in V(G_i)$, $(i = 0, \dots, d-1)$, the subgraph of $\prod_{i=0}^{d-1} G_i$ induced by $\{(v_0, \dots, v_{j-1}, u_j, v_{j+1}, \dots, v_{d-1}) \mid u_j \in V(G_{j-1})\}$ is isomorphic to G_j. We call such a subgraph a *fibre* of $\prod_{i=0}^{d-1} G_i$ or a $(j+1)$th *fibre* of $\prod_{i=0}^{d-1} G_i$ at $(v_0, \dots, v_{d-1})$. Note that we use the subscripts $0, 1, \dots, d-1$ instead of $1, \dots, d$ because it is convenient to express the elements in S_d as permutations of $0, 1, \dots, d-1$.

Recall that subgraphs of a graph G whose connected components are Eulerian graphs form, under symmetric difference and natural scalar multiplication by 0 and 1, a vector space over $\mathbb{Z}_2$ called a *cycle space* $\mathcal{C}(G)$ of G. Here a subgraph F is interpreted as a vector over $\mathbb{Z}_2$ by identifying the edge-set of F with its indicator vector. In [2], R. Hammack constructed a basis of the cycle space $\mathcal{C}(G\Box H)$ composed of some parallel 4-cycles and some cycles in each fibre of $G\Box H$. Hence we have the following proposition.

Proposition 2.1. *For given graphs $G_0, \dots, G_{d-1}$, there is a basis of the cycle space $\mathcal{C}(\prod_{i=0}^{d-1} G_i)$ composed of some parallel 4-cycles and some cycles in each fibre of $\prod_{i=0}^{d-1} G_i$.*

Let G be embedded into a nonorientable surface and let $\mathcal{B}(G)$ be a basis of cycle space $\mathcal{C}(G)$ such that $\mathcal{B}(G)$ is composed of some cycles in G. Now we have an assignment ω from $\mathcal{B}(G)$ to the 2-element field $\mathbb{Z}_2 = \{0, 1\}$, which is also a vector space over itself, such that if a cycle in $\mathcal{B}(G)$ is reversible then the assigned value is 1; otherwise 0. Now the map ω can be extended to the linear map from the cycle space $\mathcal{C}(G)$ to $\{0, 1\}$ such that for any $C_1, \dots, C_k \in \mathcal{B}(G)$, $\omega(C_1+\cdots+C_k) = \omega(C_1)+\cdots+\omega(C_k)$, where the addition in $C_1 + \cdots + C_k$ is the symmetric difference. Note that for a cycle C in G with $C = C_1 + \cdots + C_k$ for some $C_1, \dots, C_k \in \mathcal{B}(G)$, $\omega(C) = 1$ if and only if the number of reversible cycles among $C_1, \dots, C_k$ is odd, which is equivalent to that C is reversible. Hence there is a reversible cycle in $\mathcal{B}(G)$, and so we have the following corollary.

Corollary 2.2. *For given graphs $G_0, \dots, G_{d-1}$, suppose that there is a nonorientable embedding of $\prod_{i=0}^{d-1} G_i$. Then there is a reversible parallel 4-cycle or reversible cycle in some fibre of $\prod_{i=0}^{d-1} G_i$.*

For given prime graphs $G_0, \dots, G_{d-1}$, suppose that there is a nonorientable regular embedding $\mathcal{M}$ of $\prod_{i=0}^{d-1} G_i$. Then there is a graph automorphism ϕ fixing some vertex v of $\prod_{i=0}^{d-1} G_i$ and ϕ cyclically permute all neighbours of v. This implies that ϕ cyclically permute the d fibres containing v [4,11]. This means that all $G_0, \dots, G_{d-1}$ are isomorphic, namely, $\prod_{i=0}^{d-1} G_i$ is isomorphic to G_0^d. Therefore, to consider nonorientable regular embeddings of cartesian products, it suffices to consider such embeddings of G^d for a prime graph G.