Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Alternating sign matrices and their Bruhat order

Richard A. Brualdi [a,*], Michael W. Schroeder [b]

[a] Department of Mathematics, University of Wisconsin, Madison, WI 53706, United States
[b] Department of Mathematics, Marshall University, Huntington, WV 25755, United States

ARTICLE INFO

Article history:
Received 4 April 2016
Received in revised form 26 September 2016
Accepted 18 October 2016
Available online xxxx

Keywords:
Permutations
Bigrassmanians
Alternating sign matrices (ASMs)
Bruhat order

ABSTRACT

The set $\mathcal{S}_n$ of $n\times n$ permutation matrices forms a ranked partially ordered set under the Bruhat order. The Bruhat order on $\mathcal{S}_n$ can be equivalently defined by means of an entrywise partial order on an associated matrix. Lascoux and Schützenberger proved that the MacNeille completion (the unique smallest lattice containing a partially ordered set) of the Bruhat order on $\mathcal{S}_n$ is the set $\mathcal{A}_n$ of $n\times n$ alternating sign matrices (ASMs) with a partial order defined by this same entrywise order giving a ranked lattice. We continue investigations of the structure of this lattice. We show that the lattice contains some special dense ASMs defining intervals which are Boolean lattices which together span (in terms of hitting all ranks) the lattice from its minimal element to its maximal element. We also show that the number of ASMs of a given rank is a polynomial in n of degree r and obtain a natural maximal saturated chain. The Hasse diagram of a poset can be regarded as a (directed) graph, and we determine the maximal indegree, outdegree, and total degree of ASMs in $\mathcal{A}_n$; a similar determination was done by Adin and Roichman for the Bruhat order on $\mathcal{S}_n$. The join-irreducible permutations in $\mathcal{P}_n$ are the bigrassmanians, that is, permutations σ such that both σ and its inverse σ^{-1} have exactly one descent. The bigrassmanians are also the join-irreducible elements of the lattice on $\mathcal{A}_n$. We determine the minimal set of bigrassmanians whose joins are the special dense ASMs.

1. Introduction

Let $\mathcal{S}_n$ denote the symmetric group of permutations of $\{1, 2, \ldots, n\}$. We identify $\mathcal{S}_n$ with the group $\mathcal{P}_n$ of $n\times n$ permutation matrices under matrix multiplication, and we sometimes do not distinguish between the elements of $\mathcal{S}_n$ and those of $\mathcal{P}_n$. There is a partial order defined on $\mathcal{S}_n$ (and so on $\mathcal{P}_n$) which is usually called the *Bruhat order* and is denoted by $\preceq_B$. If $\sigma, \tau \in \mathcal{S}_n$, then $\sigma \preceq_B \tau$ (*σ precedes τ in the Bruhat order*) provided σ can be obtained from τ by a sequence of transpositions (k, l) (interchanging k and l in a permutation) each of which decreases the number of inversions. Thus the smallest permutation in the partially ordered set $(\mathcal{S}_n, \preceq_B)$ is the identity permutation, denoted as $\iota_n = (1, 2, \ldots, n)$, and the largest permutation is the *anti-identity permutation*, denoted as $\zeta_n = (n, n-1, \ldots, 1)$ with corresponding permutation matrix

$$L_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ & & & \cdots & & & & \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

* Corresponding author.
E-mail addresses: brualdi@math.wisc.edu (R.A. Brualdi), schroederm@marshall.edu (M.W. Schroeder).

http://dx.doi.org/10.1016/j.disc.2016.10.010

In terms of permutation matrices $P, Q \in \mathcal{P}_n$, the Bruhat order is the following:
$P \preceq_B Q$ if and only if P can be gotten from Q by sequentially replacing 2×2 submatrices equal to

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ with } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is equivalent to sequentially adding the matrix

$$\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \tag{1}$$

to any 2×2 submatrix (not necessarily of consecutive rows and columns) where the result of each addition is a permutation matrix. If σ and τ are permutations in $\mathcal{S}_n$, then τ *covers* σ in the Bruhat order if and only if $\sigma \preceq_B \tau$ and τ has exactly one more inversion than σ. Thus if $\tau = (i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_{l-1}, i_l, i_{l+1}, \ldots, i_n)$ with $k < l$ and $i_k > i_l$, and we perform the transposition which interchanges i_k and i_l, then $\sigma = (i_1, \ldots, i_{k-1}, i_l, i_{k+1}, \ldots, i_{l-1}, i_k, i_{l+1}, \ldots, i_n)$ where, in order that the number of inversions decreases by exactly one, each of $i_{k+1}, \ldots, i_{l-1}$ is either greater than i_k or less than i_l. In terms of the partially ordered set $(\mathcal{P}_n, \preceq_B)$, a permutation matrix Q covers a permutation matrix P provided P is obtained from Q by replacing a submatrix of consecutive rows and columns of Q of the form

$$\left[\begin{array}{c|ccc|c} 0 & 0 & \cdots & 0 & 1 \\ \hline 0 & & & & 0 \\ \vdots & & O & & \vdots \\ 0 & & & & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \end{array}\right]$$

with

$$\left[\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & & & & 0 \\ \vdots & & O & & \vdots \\ 0 & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

(The zero submatrix O may be vacuous.) The Hasse diagram of the Bruhat order of $\mathcal{S}_3$ is given in Fig. 1.

There is an equivalent way to determine whether or not two permutation matrices P and Q in $\mathcal{P}_n$ are related in the Bruhat order which is computationally easy since it involves only the comparison of $(n-1)^2$ pairs of integers [2,11]. For any $m \times n$ matrix $A = [a_{ij}]$, define the *sum-matrix* of A, denoted $\Sigma(A) = [\sigma_{ij}]$, as the $m \times n$ matrix where

$$\sigma_{ij} = \sigma_{ij}(A) = \sum_{k \le i} \sum_{l \le j} a_{kl} \quad (1 \le i \le m,\ 1 \le j \le n),$$

the sum of the entries in the leading $i \times j$ submatrix of A. If P is an $n \times n$ permutation matrix, then the entries in the last row and last column of $\Sigma(A)$ are $1, 2, \ldots, n$ in that order.

Lemma 1 ([2,11]). *Let $P, Q \in \mathcal{P}_n$. Then $P \preceq_B Q$ if and only if*

$$\Sigma(P) \ge \Sigma(Q) \text{ (entrywise)},$$

that is, $\Sigma(P)$ dominates $\Sigma(Q)$.

We calculate that

$$\Sigma(I_n) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}, \tag{2}$$