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Stability properties of the plethysm: A combinatorial approach

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ABSTRACT

An important family of structural constants in the theory of symmetric functions and in the representation theory of symmetric groups and general linear groups are the *plethysm coefficients*. In 1950, Foulkes observed that they have some stability properties: certain sequences of plethysm coefficients are eventually constant. Such stability properties were proven by Brion with geometric techniques, and by Thibon and Carré by means of vertex operators.

In this paper we present a new approach to prove such stability properties. Our proofs are purely combinatorial and follow the same scheme. We decompose plethysm coefficients in terms of other plethysm coefficients related to the complete homogeneous basis of symmetric functions. We show that these other plethysm coefficients count integer points in polytopes and we prove stability for them by exhibiting bijections between the corresponding sets of integer points of each polytope.

1. Introduction

The understanding of structural constants is one of the most important problems in representation theory. It is a difficult problem and even in simplest cases we can find unsolved problems.

Recall that any (finite–dimensional, complex, analytic) linear representation V of $GL_n(\mathbb{C})$ decomposes as a direct sum of irreducible representations:

$$V \approx \bigoplus m_\lambda S_\lambda(\mathbb{C}^n)$$

where the m_λ are non–negative integers called *multiplicities* and the $S_\lambda(\mathbb{C}^n)$ are the *irreducible representations of* $Gl_n(\mathbb{C})$, which are indexed by partitions of length at most n. A *partition* λ of a non-negative integer n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ and $\sum_i \lambda_i = n$. The length of a partition λ, $\ell(\lambda)$, is the number of non–zero λ_i, also called the parts of the partition. The size of a partition λ is the sum of its parts. Any $\lambda_i = 0$ is considered irrelevant and we can identify λ with the infinite sequence $(\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$. With identification, we define the sum of partitions as the sum of the corresponding sequences, and the multiplication of a partition by a scalar as the partition obtained multiplying each part of the partition by the scalar. These definitions can be extended to sequences of positive integers.

Important families of structural constants appear in non–trivial constructions of new representations from old ones. Three of them are particularly important in a more combinatorial context. First, consider the tensor product of two irreducible representations $S_\mu(\mathbb{C}^n) \otimes S_\nu(\mathbb{C}^n)$ and decompose it into irreducible. Then, the multiplicities arising from this are the

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Littlewood–Richardson coefficients, $c^{\lambda}_{\mu\nu}$. Combinatorial interpretations behind this family are well understood: for example, they count Littlewood–Richardson Young tableaux or integral points in the hive polytopes. Moreover, these interpretations are efficient tools for proofs and computations.

Next, consider an irreducible representation of $GL_{mn}(\mathbb{C})$ as a representation of $GL_m(\mathbb{C})\times GL_n(\mathbb{C})$ by means of the Kronecker product of matrices, and decompose the resulting representation into irreducible. The multiplicities arising from this operation are the *Kronecker coefficients*, $g^{\nu}_{\lambda\mu}$. In this case, combinatorial interpretations for the coefficients are known only in very particular cases. For example, C. Ballantine and R. Orellana [2] have an interpretation in terms of Kronecker tableaux of the Kronecker coefficients when one of the partitions has only two parts, and M. Rosas in [16] gives a description for the Kronecker product of Schur functions indexed by hook shapes and two-row shapes.

Finally, the plethysm coefficients $a^{\nu}_{\lambda,\mu}$ are the multiplicities obtained when we apply a Schur functor S_λ to an irreducible representation $S_\mu(\mathbb{C}^n)$ and we decompose the resulting representation into irreducible.

One of the major problems in combinatorial representation theory is to find interpretations for the Kronecker coefficients and the plethysm coefficients akin to those known for the Littlewood–Richardson coefficients. In this paper, we consider a family of plethysm coefficients and we give a combinatorial interpretation of them.

Another important problem in representation theory is to understand the stability properties that we can observe for the different kind of coefficients.

For the Kronecker case, Murnaghan [14] and Littlewood [10] observed that some sequences stabilize (they are eventually constant). We can find results about these stability phenomena in papers of E. Vallejo [19], L. Manivel [12], I. Pak and G. Panova [15], or E. Briand, R. Orellana and M. Rosas [3], for example.

The plethysm coefficients also show stability properties. In 1950, Foulkes, in [6], was the first one who observed some of those stability properties and in the 1990's we can find the first proofs of them.

We are interested in those phenomena. In this paper, we consider several sequences of plethysm coefficients. Thanks to our combinatorial interpretation of them, we give new combinatorial proofs of their stability properties.

Plethysm coefficients can be computed using the language of symmetric functions: they corresponds to the coefficients appearing in the expansion on the Schur basis $\{s_\nu\}$ of $s_\lambda[s_\mu]$, which is the operation, also called *plethysm*, induced by the plethysm of S_λ and $S_\mu(\mathbb{C}^n)$ in representation theory. Then, our results will be set in the symmetric function framework.

Before proceeding, we specify some notation we will use. We denote by $\lambda(n)$, $\mu(n)$ and $\nu(n)$ the partitions depending on n that we are considering. We say that a sequence of coefficients with general term $a^{\nu(n)}_{\lambda(n),\mu(n)}$ stabilizes if there exists $N \geq 0$ such that, for all $n \geq N$, $a^{\nu(n)}_{\lambda(n),\mu(n)} = a^{\nu(N)}_{\lambda(N),\mu(N)}$.

The following result summarizes the stability properties that we consider.

Theorem 1.1. *For any partitions λ, μ, ν and π, such that $|\lambda|\cdot|\mu| = |\nu|$, the sequences with general term*

$$a^{\nu+(n\cdot|\lambda|)}_{\lambda,\mu+(n)},\ a^{\nu+n\cdot|\lambda|\cdot\pi}_{\lambda,\mu+n\cdot\pi},\ a^{\nu+n\cdot\mu}_{\lambda+(n),\mu},\ a^{\nu+(n\cdot|\mu|)}_{\lambda+(n),\mu}$$

stabilize.

Observation. If $|\lambda|\cdot|\mu| \neq |\nu|$, the coefficient $a^{\nu}_{\lambda,\mu}$ is zero. From now, we only consider partitions that satisfy this condition.

These sequences had been studied before by other authors. The following list summarizes what has been done before.

($P1$) In [5, Theorem 4.2], it is proved that the sequence with general term

$$a^{\nu+(|\mu|\cdot n)}_{\lambda+(n),\mu} = \left\langle s_{\lambda+(n)}[s_\mu], s_{\nu+(|\mu|\cdot n)}\right\rangle$$

stabilizes, but with limit zero when $\ell(\mu) > 1$.

($Q1$) In [4, Corollary 1, Section 2.6], the sequence with general term

$$a^{\nu+n\cdot\mu}_{\lambda+(n),\mu} = \left\langle s_{\lambda+(n)}[s_\mu], s_{\nu+n\cdot\mu}\right\rangle$$

is considered as a function of $n \geq 0$ and it is proved that it is an increasing function. It is also proved that it is constant for n sufficiently large.

($R1$) In [5, Theorem 4.1], it is proved that the sequence with general term

$$a^{\nu+(|\lambda|\cdot n)}_{\lambda,\mu+(n)} = \left\langle s_\lambda[s_{\mu+(n)}], s_{\nu+(|\lambda|\cdot n)}\right\rangle$$

stabilizes for n large enough.

($R2$) In [4, Corollary 1, Section 2.6], the sequence with general term

$$a^{\nu+n\cdot|\lambda|\cdot\pi}_{\lambda,\mu+n\cdot\pi} = \left\langle s_\lambda[s_{\mu+n\cdot\pi}], s_{\nu+n\cdot|\lambda|\cdot\pi}\right\rangle$$

is considered as a function of $n \geq 0$ and it is proved that it is an increasing function. It is also proved that it is constant for n sufficiently large.

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