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Total weight choosability of graphs with bounded maximum average degree

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ABSTRACT

A *total weighting* of a graph G is a function ϕ that assigns a weight to each vertex and each edge of G. The *vertex-sum* of a vertex v with respect to ϕ is $S_\phi(v) = \phi(v) + \sum_{e\in E(v)}\phi(e)$, where $E(v)$ is the set of edges incident to v. A total weighting is *proper* if adjacent vertices have distinct vertex-sums. A graph G is (k, k')*-choosable* if the following is true: Whenever each vertex x is assigned a set $L(x)$ of k real numbers and each edge e is assigned a set $L(e)$ of k' real numbers, there is a proper total weighting ϕ of G with $\phi(y) \in L(y)$ for all $y \in V(G)\cup E(G)$. In this paper, we prove that for $p \in \{5, 7, 11\}$, a graph G without isolated edges and with $\text{mad}(G) \le p-1$ is $(1, p)$-choosable. In particular, triangle-free planar graphs are $(1, 5)$-choosable.

1. Introduction

Suppose $G = (V, E)$ is a graph. For each $z \in V(G)\cup E(G)$, let x_z be a variable associated to z. Fix an orientation D of G. Define polynomial $P_G(\{x_z : z \in V(G)\cup E(G)\})$ as follows:

$$P_G(\{x_z : z \in V(G)\cup E(G)\}) = \prod_{uv\in E(D)}\left(\left(\sum_{e\in E_G(v)} x_e + x_v\right) - \left(\sum_{e\in E_G(u)} x_e + x_u\right)\right),$$

where $E_G(v)$ is the set of edges incident to v. For each $z \in V(G)\cup E(G)$, assign a real number $\phi(z)$ to the variable x_z, and view $\phi(z)$ as the weight of z. Let $P_G(\phi)$ be the evaluation of the polynomial at $x_z = \phi(z)$. We say ϕ is a *proper total weighting* of G if $P_G(\phi) \neq 0$. In other words, ϕ is a proper total weighting of G if for any two adjacent vertices u and v, $\sum_{e\in E(u)}\phi(e) + \phi(u) \neq \sum_{e\in E(v)}\phi(e) + \phi(v)$.

A mapping $\eta : V(G)\cup E(G) \to \{0, 1, \ldots\}$ is called an *index function of* G. An index function η is valid if $\sum_{z\in V(G)\cup E(G)}\eta(z) = |E(G)|$. We say G is η*-choosable* if for any list assignment L which assigns to $z \in V(G)\cup E(G)$ a set $L(z)$ of $\eta(z)$ real numbers as permissible weights, there is a proper total weighting ϕ of G with $\phi(z) \in L(z)$ for all $z \in V(G)\cup E(G)$. It follows from the Combinatorial Nullstellensatz that if η' is valid index function and the coefficient $c_{\eta',G}$ of the monomial $\prod_{z\in V(G)\cup E(G)} x_z^{\eta'(z)}$ in the expansion of P_G is nonzero, and $\eta(z) > \eta'(z)$ for all z, then G is η-choosable. This motivates the following definition.

Definition 1. For an index function η of G, if there is a valid index function η' of G for which $\eta'(z) < \eta(z)$ for all $z \in V(G)\cup E(G)$ and $c_{\eta',G} \neq 0$, then we say G is *strongly* η*-choosable*. Here $c_{\eta',G}$ is the coefficient of the monomial $\prod_{z\in V(G)\cup E(G)} x_z^{\eta'(z)}$ in the expansion of P_G.

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So strongly η-choosable means "η-choosable that can be proved by using Combinatorial Nullstellensatz" [3,4]. We say a graph is (strongly) (k, k')-choosable if it is (strongly) η-choosable, where $\eta(v) = k$ for each vertex v and $\eta(e) = k'$ for each edge e. Note that $(k, 1)$-choosable is equivalent to vertex k-choosable.

The well-known 1–2–3 conjecture [10] says that any graph G with no isolated edges has a proper total weighting ϕ with $\phi(v) = 0$ for all vertices v, and $\phi(e) \in \{1, 2, 3\}$ for all edges e. As a strengthening of the 1-2-3-conjecture, it was proposed in [22] that every graph with no isolated edges is (strongly) $(1, 3)$-choosable.

There are many partial results on the 1-2-3 conjecture and on the total weight choosability conjectures [1,2,5–7,9–23]. It was shown in [23] that every graph is $(2, 3)$-choosable. It was proved in [9] that every graph with no isolated edge has a vertex-colouring 5-edge weighting. However, it is unknown whether there is a constant k such that every graph with no isolated edge is $(1, k)$-choosable. It is also unknown whether there is a constant k such that every graph is $(k, 2)$-choosable.

For sparse graphs, it was proved in [7] that 1-2-3 conjecture and 1-2 conjecture hold for graphs with $\mathrm{mad}(G) < \frac{8}{3}$. For graphs with $\mathrm{mad}(G) < \frac{5}{2}$, it was proved in [14] that $(1, 3)$-choosable conjecture holds for some special list assignments (namely, for those L with $L(v) = \{0\}$ for each vertex v and $L(e)$ contains three *positive* numbers for each edge e) and $(2, 2)$-choosable conjecture holds. Recently, the following result was proved in [8] that every graph G without isolated edges is $(1, \Delta(G) + 1)$-choosable. It was proved in [11] that a graph G without an isolated edge and with $\mathrm{mad}(G) < \frac{11}{4}$ is strongly $(1, 3)$-choosable.

In this paper, we prove that a graph G without isolated edges and $\mathrm{mad}(G) \le p - 1$ is strongly $(1, p)$-choosable, where $p \in \{5, 7, 11\}$. In particular, triangle-free planar graphs are $(1, 5)$-choosable.

2. Preliminaries

The polynomial P_G can be written as

$$P_G = \prod_{uv\in E(D)} \sum_{z\in V(G)\cup E(G)} A_G[e, z]x_z,$$

where for $e = uv \in E(D)$ and $z \in V(G) \cup E(G)$,

$$A_G[uv, z] = \begin{cases} 1 & \text{if } z \in \{v\} \cup (E(v) - \{uv\}), \\ -1 & \text{if } z \in \{u\} \cup (E(u) - \{uv\}), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that the matrix A_G depends on the orientation D of G. However, reversing an edge simply multiplies the corresponding row by -1 and will not affect whether $c_{\eta,G}$ is nonzero. So for convenience, we ignore the orientation in the notation. For $z \in V(G) \cup E(G)$, let $A_G(z)$ denote the column of A_G indexed by z.

For an index function η of G, let $A_G(\eta)$ be the matrix consisting $\eta(z)$ copies of column $A_G(z)$ for each $z \in V \cup E$.

It is well-known (cf. [22]) that for a valid index function η of G,

$$c_{\eta,G} = \frac{1}{\prod_{z\in V(G)\cup E(G)} \eta(z)!}\mathrm{per}(A_G(\eta)),$$

where $\mathrm{per}(A)$ is the permanent of A. Thus G is strongly η-choosable if and only if there is a valid index function η' with $\eta'(z) < \eta(z)$ for all $z \in V(G) \cup E(G)$, and for which $\mathrm{per}(A_G(\eta')) \ne 0$. We call such a matrix $A_G(\eta')$ a *witness* of G being strongly η-choosable.

Assume M is a square matrix whose columns are linear combinations of columns of A_G. Define an index function $\eta_M : V(G) \cup E(G) \to \{0, 1, \ldots\}$ as follows: For $z \in V(G) \cup E(G)$, $\eta_M(z)$ is the number of columns of M in which $A_G(z)$ appears with nonzero coefficient.

The columns of A_G are not linearly independent. Indeed, it is well-known [18] and easy to verify that for an edge $e = uv$ of G, we have

$$A_G(e) = A_G(u) + A_G(v). \tag{2}$$

Thus η_M is not determined by the matrix M itself, but depends on how the columns of M are expressed as linear combinations of columns of A_G. Whenever the index function η_M is used, explicit expressions of the columns of M as linear combinations of columns of A_G are given. The function η_M refers to those particular expressions of linear combinations. For brevity, the linear combinations are not shown in the notation η_M.

As the permanent of a matrix is linear with respect to its column vectors, if M is a matrix whose columns are linear combinations of columns of A_G, such that $\mathrm{per}(M) \ne 0$, then there is a valid index function η of G with $\eta(z) \le \eta_M(z)$ for all $z \in V(G) \cup E(G)$, and $\mathrm{per}(A_G(\eta)) \ne 0$. Thus to prove a graph G is strongly η-choosable, it suffices to show that there is a matrix M whose columns are linear combinations of columns of A_G, such that $\mathrm{per}(M) \ne 0$ and $\eta_M(z) < \eta(z)$ for all $z \in V(G) \cup E(G)$.

The following lemmas proved in [21] will be needed in our proofs. By an edge column of A_G, we mean a column of the form $A_G(e)$ for an edge $e \in E(G)$. A vertex column of A_G is defined similarly.

Lemma 1 ([21]). *Assume p is a prime number and M is a matrix whose columns are integral linear combinations of edge columns of A_G with $\mathrm{per}(M) \ne 0 \pmod p$. Then G is strongly $(1, p)$-choosable.*