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Set systems with positive intersection sizes

Jiuqiang Liu [a,b], Xiaodong Liu [a]

[a] School of Management Engineering, Xi'an University of Finance and Economics, Xi'an Shaanxi, 710100, PR China
[b] Department of Mathematics, Eastern Michigan University, Ypsilanti, MI 48197, USA

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ABSTRACT

In this paper, we derive a best possible k-wise extension to the well-known Snevily theorem on set systems (Snevily, 2003) which strengthens the well-known theorem by Füredi and Sudakov (2004). We also provide a conjecture which gives a common generalization to all existing non-modular $\mathcal{L}$-intersection theorems.

1. Introduction

A family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called *intersecting* if every pair of distinct subsets $E, F \in \mathcal{F}$ have a nonempty intersection. Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. A family $\mathcal{F}$ of subsets of $[n]$ is called k-wise $\mathcal{L}$*-intersecting* if $|F_1 \cap F_2 \cap \cdots \cap F_k| \in \mathcal{L}$ for every collection of k distinct subsets in $\mathcal{F}$. When $k = 2$, such a family $\mathcal{F}$ is called $\mathcal{L}$*-intersecting*. $\mathcal{F}$ is r*-uniform* if it is a collection of r-subsets of $[n]$. Thus, a r-uniform intersecting family is $\mathcal{L}$-intersecting for $\mathcal{L} = \{1, 2, \ldots, r-1\}$.

In 1961, Erdös, Ko, and Rado [2] proved the following classical result.

Theorem 1.1 (*Erdös, Ko, and Rado, 1961 [2]*). *Let $n \geq 2k$ and let $\mathcal{A}$ be a k-uniform intersecting family of subsets of $[n]$. Then, $|\mathcal{A}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{A}$ consists of all k-subsets containing a common element.*

To date, many intersection theorems have appeared in the literature, see [8] for a brief survey on theorems about $\mathcal{L}$-intersecting families. Here are well-known Frankl–Wilson theorem [5] and Alon–Babai–Suzuki theorem [1].

Theorem 1.2 (*Frankl and Wilson, 1981*). *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. If $\mathcal{A}$ is an $\mathcal{L}$-intersecting family of subsets of $[n]$, then*

$$|\mathcal{A}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

Theorem 1.3 (*Alon, Babai, and Suzuki, 1991*). *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers and $K = \{k_1, k_2, \ldots, k_r\}$ be a set of integers satisfying $k_i > s - r$ for every i. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is a family of subsets of $[n]$ such that*

E-mail address: jliu@emich.edu (J. Liu).

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$|A_i| \in K$ for every $1 \le i \le m$ and $|A_i \cap A_j| \in \mathcal{L}$ for every pair $i \ne j$. Then,

$$m \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

Stronger bounds can be obtained if information about the specific set $\mathcal{L}$ is used. To that end, Snevily [12] proved in 2003 the following theorem conjectured in 1994 by himself [10], which provides a common generalization of Frankl–Füredi theorem [4] (where $\mathcal{L} = \{1, 2, \ldots, s\}$) and Frankl–Wilson theorem (Theorem 1.2).

Theorem 1.4 (*Snevily, 2003*). *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers. If $\mathcal{A}$ is an $\mathcal{L}$-intersecting family of subsets of $[n]$, then*

$$|\mathcal{A}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}.$$

We will derive the following asymptotically best possible k-wise extension of Theorem 1.4.

Theorem 1.5. *Let $k \ge 2$ and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers. If $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is a family of subsets of $[n]$ such that $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \in \mathcal{L}$ for every collection of k distinct subsets in $\mathcal{A}$, then there exists $n_0 = n_0(k, s)$ such that for all $n \ge n_0$*

$$m \le \frac{k+s-1}{s+1}\binom{n-1}{s} + \sum_{i \le s-1} \binom{n-1}{i}.$$

Clearly, Theorem 1.5 implies the following well-known theorem by Füredi and Sudakov [6]: simply add a new common element $n+1$ to every subset in the family which satisfies the condition in the next theorem.

Theorem 1.6 (*Füredi and Sudakov, 2004*). *Let $k \ge 2$ and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. If $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is a family of subsets of $[n]$ such that $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \in \mathcal{L}$ for every collection of k distinct subsets in $\mathcal{A}$, then there exists $n_0 = n_0(k, s)$ such that for all $n \ge n_0$*

$$m \le \frac{k+s-1}{s+1}\binom{n}{s} + \sum_{i \le s-1} \binom{n}{i}.$$

We remark that the bound in Theorem 1.5 is asymptotically the best possible, as shown by the following example: For $k \ge 3$, by applying Lemma 2.2 in [6] on the $(n-1)$-element set $[n-1] = \{1, 2, \ldots, n-1\}$ and then adding the new common element n to every subset in the family, we obtain a k-wise $\mathcal{L}$-intersecting family $\mathcal{A}$ of subsets of $[n]$ with $\mathcal{L} = \{1, 2, \ldots, s\}$ such that

$$|\mathcal{A}| \ge \frac{k-2}{s+1}\left(1 - \frac{s}{n}\right)\binom{n-1}{s} + \sum_{i \le s} \binom{n-1}{i} = \frac{k+s-1}{s+1}\left(1 - \frac{s}{n}\right)\binom{n-1}{s} + \sum_{i \le s-1} \binom{n-1}{i}.$$

The next conjecture provides a common generalization to all existing non-modular $\mathcal{L}$-intersection theorems in the literature if it is true.

Conjecture 1.7. *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers with $l_1 < l_2 < \cdots < l_s$ and $K = \{k_1, k_2, \ldots, k_r\}$ be a set of integers satisfying $k_i > s - r$ for every i. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is a family of subsets of $[n]$ such that $|A_i| \in K$ for every $1 \le i \le m$ and $|A_i \cap A_j| \in \mathcal{L}$ for every pair $i \ne j$. Then,*

$$m \le \binom{n-l_1}{s} + \binom{n-l_1}{s-1} + \cdots + \binom{n-l_1}{s-r+1}.$$

The classical Erdös–Ko–Rado theorem (Theorem 1.1) is the special case of Conjecture 1.7 with $l_1 \ge 1$, $r = 1$, and $\mathcal{L} = \{1, 2, \ldots, k-1\}$; the well-known theorem for t-intersecting family of k-subsets of $[n]$ by Erdös, Ko, and Rado [2], Frankl [3], and Wilson [13] is the special case with $l_1 = t$, $r = 1$, and $\mathcal{L} = \{t, t+1, \ldots, k-1\}$; the Frankl–Wilson Theorem (Theorem 1.2) is the special case $l_1 \ge 0$ and $r = n$; the well-known Alon–Babai–Suzuki theorem (Theorem 1.3) is the special case with $l_1 \ge 0$; the Snevily theorem (Theorem 1.4) is the special case with $l_1 \ge 1$ and $r = n$; and the well-known Ray–Chaudhuri–Wilson theorem [9] is the special case with $l_1 \ge 0$ and $r = 1$. The bound in the conjecture is the best possible, as shown by the family of all subsets of $[n]$ with sizes at most $s + l_1$ and at least $s - r + 1 + l_1$ which contain all $1, 2, \ldots, l_1$, where $\mathcal{L} = \{l_1, l_1 + 1, \ldots, s + l_1 - 1\}$.

2. Preliminaries

Throughout this section, we denote $[n] = \{1, 2, \ldots, n\}$ and use $x = (x_1, x_2, \ldots, x_n)$ to denote a vector of n variables with each variable x_j taking values 0 or 1. A polynomial $f(x)$ in n variables x_i, $1 \le i \le n$, is called *multilinear* if the power of