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# 3-Restricted arc connectivity of digraphs[☆]

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ABSTRACT

The $k$-restricted arc connectivity of digraphs is a common generalization of the arc connectivity and the restricted arc connectivity. An arc subset $S$ of a strong digraph $D$ is a $k$-restricted arc cut if $D - S$ has a strong component $D'$ with order at least $k$ such that $D - V(D')$ contains a connected subdigraph with order at least $k$. The $k$-restricted arc connectivity $\lambda^k(D)$ of a digraph $D$ is the minimum cardinality over all $k$-restricted arc cuts of $D$.

Let $D$ be a strong digraph with order $n \geq 6$ and minimum degree $\delta(D)$. In this paper, we first show that $\lambda^3(D)$ exists if $\delta(D) \geq 3$ and, furthermore, $\lambda^3(D) \leq \xi^3(D)$ if $\delta(D) \geq 4$, where $\xi^3(D)$ is the minimum 3-degree of $D$. Next, we prove that $\lambda^3(D) = \xi^3(D)$ if $\delta(D) \geq \frac{n+3}{2}$. Finally, we give examples showing that these results are best possible in some sense.



## 1. Terminology and introduction

For graph-theoretical terminology and notation not defined here we follow [4]. We only consider finite digraphs without loops and multiple arcs. The set of vertices and the set of arcs of a digraph $D$ are denoted by $V(D)$ and $A(D)$, respectively. In a digraph $D$, the *out-neighborhood* of a vertex $x$ is the set $N_D^+(x) = \{v \in V(D) : xv \in A(D)\}$ and the *out-degree* of $x$ is $d_D^+(x) = |N^+(x)|$. The *in-neighborhood* and *in-degree* of $x$ are defined analogously. The *minimum degree* of $D$ is $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$, where $\delta^+(D) = \min\{d^+(x) : x \in V(D)\}$ is the *minimum out-degree* of $D$ and $\delta^-(D) = \min\{d^-(x) : x \in V(D)\}$ is the *minimum in-degree* of $D$.

For a pair $X, Y$ of nonempty vertex sets of a digraph $D$, we define $A(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. For brevity, we denote $\overline{X} = V(D) \setminus X$. If $Y = \overline{X}$, we write $\partial_D^+(X)$ or $\partial_D^-(Y)$ instead of $A(X, Y)$. In this paper, we will not distinguish a set of cardinality 1 from its only element. For example, the arc set $\partial_D^+(\{x\})$ will be abbreviated to $\partial_D^+(x)$. Clearly, $d_D^+(x) = |\partial_D^+(x)|$ for all $x \in V(D)$. We will usually omit the subscript for the digraph when it is clear which digraph is meant.

For $X \subseteq V(D)$, *the subdigraph of D induced by X* is denoted by $D[X]$. The *underlying graph* $UG(D)$ of a digraph $D$ is the unique graph obtained from $D$ by deleting the orientation of all arcs and keeping one edge of a pair of multiple edges. A digraph $D$ is *connected* if its underlying graph $UG(D)$ is connected. A digraph $D$ is *strongly connected* (or, just, *strong*) if, for every pair $x, y$ of distinct vertices in $D$, there exists a directed $(x, y)$-path and a directed $(y, x)$-path. We define a digraph with one vertex to be strong. A *connected (strong) component* of a digraph $D$ is a maximal induced subdigraph of $D$ which is connected (strong). If $D$ has $p$ strong components, then these strong components can be labeled $D_1, \ldots, D_p$ such that there is no arc from $D_j$ to $D_i$ unless $j < i$ [4]. We call such an ordering *an acyclic ordering of the strong components of D*. A strong component of $D$ that has no entering (leaving) arcs is called *an initial (terminal) strong component of D*.

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The underlying topology of a processor interconnection network or a communications network can be modeled by a digraph, in which the vertices represent nodes such as processors or stations, and the arcs represent links between the nodes. Therefore, the reliability of a network can be measured by the arc connectivity of a digraph. For a digraph $D$, an arc set $S$ of $D$ is an *arc cut*, if $D - S$ is not strong. The *arc connectivity* $\lambda(D)$ of $D$ is the minimum cardinality of all arc cuts of $D$.

It is well known that $\lambda(D) \leq \delta(D)$ for a general digraph $D$. A strong digraph $D$ with $\lambda(D) = \delta(D)$ is said to be $\lambda$*-optimal*. Many sufficient conditions for a digraph to be $\lambda$-optimal were presented, including the following. (For more results in this direction, see the survey by Hellwig and Volkmann [11].)

**Theorem 1.1** ([8]). *If $D$ is a strong digraph with order $n$ and minimum degree $\delta(D) \geq \frac{n-1}{2}$, then $D$ is $\lambda$-optimal.*

As a more refined index than the arc connectivity, Volkmann [12] introduced the concept of restricted arc connectivity. Let $D$ be a strong digraph. An arc subset $S$ is a *restricted arc cut* of $D$ if $D - S$ has a strong component $D_1$ with order at least 2 such that $D - V(D_1)$ contains an arc. If such an arc cut exists, then $D$ is called a $\lambda'$*-connected* digraph and the *restricted arc connectivity* of $D$, denoted by $\lambda'(D)$, is defined to be the minimum cardinality over all restricted arc cuts of $D$.

In order to give an upper bound on restricted arc connectivity, Wang and Lin [13] introduced the notion of minimum arc-degree. Let $xy$ be an arc of $D$ and let

$$\Omega(xy) = \Omega_{xy} = \{\partial^+(\{x, y\}), \partial^-(\{x, y\}), \partial^-(x) \cup \partial^+(y), \partial^+(x) \cup \partial^-(y)\}.$$

The *arc-degree of the arc $xy$* is defined as $\xi'(xy) = \min\{|S| : S \in \Omega_{xy}\}$ and the *minimum arc-degree of $D$* is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$. In [13], the following result is shown.

**Theorem 1.2** ([13]). *Let $D$ be a strong digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then, $D$ is $\lambda'$-connected and $\lambda'(D) \leq \xi'(D)$.*

In 2013, Balbuena et al. [3] proved a similar result.

**Theorem 1.3** ([3]). *Let $D$ be a strong digraph of order at least 4 and minimum degree $\delta(D) \geq 2$. Then, $D$ is $\lambda'$-connected and $\lambda'(D) \leq \xi'(D)$.*

In [13], the concept of $\lambda'$-optimality was introduced. A $\lambda'$-connected digraph $D$ is $\lambda'$*-optimal* if $\lambda'(D) = \xi'(D)$. Some sufficient conditions for digraphs to attain $\lambda'$-optimality can be found in [1,2,5,6,9,10,13]. The following result given in [13] is the first one in this direction: Let $D$ be a strong digraph with order at least 4. If $|N^+(u) \cap N^-(v)| \geq 3$ for all pairs of vertices $u, v$ with $uv \notin A(D)$, then $D$ is $\lambda'$-optimal. An immediate consequence of this result is the following.

**Corollary 1.1.** *If $D$ is a strong digraph with order $n \geq 4$ and minimum degree $\delta(D) \geq \frac{n+1}{2}$, then $D$ is $\lambda'$-optimal.*

Motivated by the $k$-restricted edge connectivity of undirected graphs, which is introduced by Fàbrega and Fiol [7], we introduce the following concept.

**Definition 1.1.** Let $D$ be a strong digraph. An arc subset $S$ is a $k$-restricted arc cut of $D$ if $D - S$ has a strong component $D'$ with order at least $k$ such that $D - V(D')$ contains a connected subdigraph with order at least $k$. If such a $k$-restricted arc cut exists, then $D$ is called $\lambda^k$-connected. The $k$-restricted arc connectivity $\lambda^k(D)$ of a $\lambda^k$-connected digraph $D$ is the minimum cardinality over all $k$-restricted arc cuts.

Clearly, $\lambda^1(D) = \lambda(D)$ and $\lambda^2(D) = \lambda'(D)$. Therefore, the $k$-restricted arc connectivity of digraphs is a common generalization of the arc connectivity and the restricted arc connectivity.

The minimum degree and the minimum arc degree of a digraph can be extended as follows. Let $D$ be a digraph and let $k$ be a positive integer. For any $X \subseteq V(D)$, let

$$\Omega(X) = \{\partial^+(X_1) \cup \partial^-(X \setminus X_1) : X_1 \subseteq X\}$$

and

$$\xi(X) = \min\{|S| : S \in \Omega(X)\}.$$

Define the *minimum $k$-degree of $D$* to be

$$\xi^k(D) = \min\{\xi(X) : X \subseteq V(D), |X| = k, D[X] \text{ is connected}\}.$$

Clearly, $\xi^1(D) = \delta(D)$ and $\xi^2(D) = \xi'(D)$. In Section 2, we prove the following result, which is a generalization of Theorem 1.2.

**Theorem 1.4.** *Let $D$ be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$. Then, $D$ is $\lambda^k$-connected and $\lambda^k(D) \leq \xi^k(D)$.*

Inspired by Theorem 1.3, we prove the following results in Sections 2 and 3, respectively.

**Theorem 1.5.** *Let $D$ be a strong digraph with order at least 6. If $\delta(D) \geq 3$, then $D$ is $\lambda^3$-connected.*

**Theorem 1.6.** *Let $D$ be a strong digraph with $|V(D)| \geq 6$ and $\delta(D) \geq 4$. Then, $D$ is $\lambda^3$-connected and satisfies $\lambda^3(D) \leq \xi^3(D)$.*