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A divide-and-conquer bound for aggregate's quality and algebraic connectivity

Artem Napov

Université Libre de Bruxelles, Service de Métrologie Nucléaire (C.P. 165/84), 50, Av. F.D. Roosevelt, B-1050 Brussels, Belgium

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ABSTRACT

We establish a divide-and-conquer bound for the aggregate's quality and algebraic connectivity measures, as defined for weighted undirected graphs. Aggregate's quality is defined on a set of vertices and, in the context of aggregation-based multigrid methods, it measures how well this set of vertices is represented by a single vertex. On the other hand, algebraic connectivity is defined on a graph, and measures how well this graph is connected. The considered divide-and-conquer bound for aggregate's quality relates the aggregate's quality of a union of two disjoint sets of vertices to the aggregate's quality of the two sets. Likewise, the bound for algebraic connectivity relates the algebraic connectivity of the graph induced by a union of two disjoint sets of vertices to the algebraic connectivity of the graphs induced by the two sets.

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1. Introduction

We establish a divide-and-conquer bound for aggregate's quality and algebraic connectivity, as defined for weighted undirected graphs. Aggregate's quality is used in the context of aggregation-based multigrid methods [2,3,13] as a tool for the design of robust multigrid solvers. Although initially introduced for discretized partial differential equations [9–11,14], aggregate's quality is now also used for graph Laplacian systems [12]. Aggregate's quality is defined on a set of vertices, also called aggregate, and measures the maximal impact on the multigrid convergence of representing this set of vertices by a single vertex.

Regarding algebraic connectivity, it has been first introduced by Fiedler in [5] as a measure of graph's connectivity, and is extensively studied ever since (see [4,6–8] for a review). For a given graph, algebraic connectivity is defined as a second smallest eigenvalue of the corresponding graph Laplacian (the first eigenvalue is always zero).

The divide-and-conquer bound for aggregate's quality and algebraic connectivity is the main contribution of this paper. In particular, the bound for aggregate's quality relates the aggregate's quality of a union of two disjoint sets of vertices to the largest aggregate's quality of individual sets. More precisely, and using the terms that are defined latter in the text, let $G = (V, E)$ be a weighted undirected graph and $S_1,\ S_2 \subset V$ be two nonempty disjoint subsets of its vertices; denoting by $\mu = \mu_G(S_1 \cup S_2)$, $\mu_1 = \mu_G(S_1)$ and $\mu_2 = \mu_G(S_2)$ the aggregation quality of the sets $S_1 \cup S_2$, S_1 and S_2, respectively, the corresponding divide-and-conquer bound is

$$\mu_c \le \mu \le \mu_c \ \max(\mu_1, \mu_2), \tag{1}$$

where μ_c is an easily-to-compute function of the weights of the graph G. The quantity μ_c can be seen as an aggregate's quality of a couple of vertices, which is obtained by shrinking S_1 and S_2 each to a vertex. A similar divide-and-conquer bound

E-mail address: anapov@ulb.ac.be.

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holds for algebraic connectivity of the subgraphs induced by $S_1 \cup S_2$, S_1 and S_2; this bound is stated with the double inequality (13) latter in the text.

The remainder of this paper is structured as follows. In Section 2 we define aggregate's quality and algebraic connectivity, and further establish some relations between them. Our main contribution – the divide-and-conquer bound – is established in Section 3. More precisely, we first obtain the divide-and-conquer bound for algebraic connectivity from the corresponding bound (1) for aggregate's quality, and then proof this latter result. The section ends with a discussion of the accuracy of the resulting bound and with a conjectured generalization of (1). Eventually, we review some potential applications in Section 4.

Notation

For any ordered set S, $S(i)$ is its ith element and $|S|$ is its size. For any subset C of the set S, $S \setminus C$ is the set of all elements that are in S and not in C. The vector $\mathbf{0}$ is a zero vector, $\mathbf{1}$ is a vector of ones, and I is an identity matrix.

2. Aggregate's quality and algebraic connectivity

The divide-and-conquer bound is established in the context of weighted undirected graphs. We therefore first recall some definitions and results from graph theory, and then introduce the aggregate's quality and algebraic connectivity measures, as well as some relations between them.

Note that the case of one vertex set and/or graph is covered in the text. This is because one vertex case arises naturally in the recursive application of divide-and-conquer bounds.

2.1. Graphs

A *simple graph* $G = (V, E)$ is defined by a set V of vertices and a set $E \subset V \times V$ of edges. G is *undirected* if $(i, j) \in E$ implies $(j, i) \in E$ for every $i, j \in V$. A simple undirected graph is *weighted* if it has an associated positive weight function $w : E \mapsto \mathbb{R}^+$ such that $w(i, j) = w(j, i)$ for every $(i, j) \in E$. In what follows, the considered graphs are assumed to be undirected and weighted.

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A subgraph $G' = (V', E')$ is *induced* by V' if E' contains all the edges in E with endpoints in V'; such a subgraph is denoted here by $G(V')$.

Both aggregate's quality and algebraic connectivity are defined with the help of the Laplacian matrix. The *Laplacian matrix* of a graph $G = (V, E)$ (or its graph Laplacian) is the $|V| \times |V|$ matrix $L(G) = (\ell_{ij})$ such that its off-diagonal entries ($i \neq j$) satisfy

$$\ell_{ij} = \begin{cases} -w(i, j) & \text{if } (i, j) \in E, \\ 0 & \text{otherwise,} \end{cases}$$

whereas the diagonal entries are set up in such a way that $L(G)$ has zero row-sum:

$$\ell_{ii} = -\sum_{j \neq i} \ell_{ij};$$

that is, such that $L(G)\mathbf{1} = \mathbf{0}$.

The null space of the Laplacian matrix $L(G)$ is spanned by $\mathbf{1}$ if and only if the corresponding graph G is *connected*, that is, if and only if there is a path of edges between any two vertices. If G is not connected, it can be decomposed into $m > 1$ connected subgraphs, called *connected components*, such that there is no connection between any two such subgraphs. In this case, the null space of $L(G)$ is spanned by the m vectors, each of which is 1 on the vertices of one connected component and 0 elsewhere.

2.2. Aggregate's quality

The aggregate's quality is defined for a nonempty subset $S \subset V$ of vertices of a graph $G = (V, E)$. Before we state its definition, we first define the matrix $\Sigma_G(S)$ as being a nonnegative diagonal matrix whose ith diagonal entry is given by

$$\sum_{j \in V \setminus S} w(S(i), j);$$

that is, by the sum of weights of edges connecting the ith vertex of S, denoted by $S(i)$, with the vertices outside S. In other words, $\Sigma_G(S)$ accounts for the connectivity of each vertex in S with the vertices of G exterior to S.

The definition of *aggregate's quality* $\mu_G(S)$ of the subset S with respect to the graph G depends on a number of conditions. If the S has only one vertex (i.e., $|S| = 1$), then we set $\mu_G(S) = 1$. If S has more than one vertex (i.e., $|S| > 1$), the definition depends on whether

(1) the subgraph $G(S)$ of G induced by S is connected;

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