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Chromatic number via Turán number

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ABSTRACT

For a graph G and a family of graphs $\mathcal{F}$, the general Kneser graph $\mathrm{KG}(G, \mathcal{F})$ is a graph with the vertex set consisting of all subgraphs of G isomorphic to some member of $\mathcal{F}$ and two vertices are adjacent if their corresponding subgraphs are edge disjoint. In this paper, we introduce some generalizations of Turán number of graphs. In view of these generalizations, we give some lower and upper bounds for the chromatic number of general Kneser graphs $\mathrm{KG}(G, \mathcal{F})$. Using these bounds, we determine the chromatic number of some family of general Kneser graphs $\mathrm{KG}(G, \mathcal{F})$ in terms of generalized Turán number of graphs. In particular, we determine the chromatic number of every Kneser multigraph $\mathrm{KG}(G, \mathcal{F})$ where G is a multigraph each of whose edges has the multiplicity at least 2 and $\mathcal{F}$ is an arbitrary family of simple graphs. Moreover, the chromatic number of general Kneser graph $\mathrm{KG}(G, \mathcal{F})$ is exactly determined where G is a dense graph and $\mathcal{F} = \{K_{1,2}\}$.

1. Introduction

Hereafter, the symbol $[n]$ stands for the set $\{1, \ldots, n\}$. A (*multi*) *hypergraph* $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite nonempty set, called the set of *vertices* of $\mathcal{H}$, and $E(\mathcal{H})$ is a nonempty family of nonempty subsets of $V(\mathcal{H})$, called the set of *hyperedges* of $\mathcal{H}$. The *multiplicity* of a hyperedge e is the number of multiple hyperedges containing the same vertices as e.

For two hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$, a *homomorphism* from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a map from the vertex set of $\mathcal{H}_1$ to that of $\mathcal{H}_2$ such that the image of any hyperedge of $\mathcal{H}_1$ contains some hyperedges of $\mathcal{H}_2$. A *t-coloring* of $\mathcal{H}$ is a mapping $h : V(\mathcal{H}) \longrightarrow [t]$ with no monochromatic hyperedge. The *chromatic number* of $\mathcal{H}$, denoted $\chi(\mathcal{H})$, is the least positive integer t (the number of colors) for which there is a t-coloring of $\mathcal{H}$. If $\mathcal{H}$ has some hyperedge of size 1, then we define its chromatic number to be infinite.

A subset $S \subseteq [n]$ is called *s-stable* if any two distinct elements of S are at least "at distance s apart" on the n-cycle, that is, $s \le |i-j| \le n-s$ for distinct $i, j \in S$. For a subset $A \subseteq [n]$, the symbols $\binom{A}{k}$ and $\binom{A}{k}_s$ stand for the set of all k-subsets of A and the set of all s-stable k-subsets of A, respectively. Furthermore, the notations C_n, K_n, $K_{m,n}$, and rK_2 stand for the n-*cycle*, the *complete graph* with n vertices, the complete bipartite graph with parts of size m and n, and the *matching* of size r, respectively. The *circular complete graph* $K_{\frac{n}{d}}$ has $[n]$ as the vertex set and two vertices i and j are adjacent if $d \le |i-j| \le n-d$. Circular complete graphs can be considered as a generalization of complete graphs and they have been studied in the literature, see [23].

Overview and Plan. In this paper, we investigate the chromatic number of graphs through their Kneser representations. A *Kneser representation* for a graph G is a bijective assignment of the hyperedges of a hypergraph $\mathcal{H}$ to the vertices of G such

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that two vertices of G are adjacent if their corresponding hyperedges are disjoint. In [1], in view of Kneser representations of graphs, the present authors introduced some lower bounds for their chromatic numbers. In this regard, for a graph G and for any $1 \leq i \leq \chi(G)$, the *ith altermatic number* of G was defined as a lower bound for the chromatic number of G. Also, it was shown that the ith altermatic number provides a tight lower bound for the chromatic number. In view of the first and second altermatic number of graphs, we here introduce the altermatic and strong altermatic number of graphs as two lower bounds for the chromatic number of graphs. It will be proved that computing each of these parameters is an NP-hard problem. Although each of the problems of computing the altermatic number of graphs, computing the strong altermatic number of graphs, and computing the chromatic number of graphs is an NP-hard problems in general, for a special family of graphs, as we will see in the paper, we might be able to compute the altermatic number or the strong altermatic number of graphs which results in computing the chromatic number of them.

This paper is organized as follows. In the rest of this section, we introduce general Kneser graphs and generalized Turán number of graphs. Several famous results and questions will be reformulated in terms of generalized Turán number of graphs. Also, a lower bound for the chromatic number of graphs will be presented in terms of generalized Turán number of graphs, see Lemma 1. In the second section, we will show that there is a tight relationship between the chromatic number of graphs and the generalized Turán number of graphs. In this regard, the chromatic number of two families of general Kneser graphs, namely the Kneser multigraph and path graphs, will be determined. In Section 3, we end the paper by proving that the altermatic number and the strong altermatic number of graphs provide two lower bounds for the chromatic number of graph, resulting in Lemma 1. Also, the complexity of computing the altermatic and strong altermatic number of graphs will be discussed.

1.1. General Kneser graphs

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph and $r \geq 2$ be an integer. The *general Kneser hypergraph* $\mathrm{KG}^r(\mathcal{H})$ is an r-uniform hypergraph whose vertex set is $E(\mathcal{H})$ and whose hyperedge set consists of all r-tuples of pairwise disjoint hyperedges of $\mathcal{H}$, i.e., $V(\mathrm{KG}^r(\mathcal{H})) = E(\mathcal{H})$ and

$$E(\mathrm{KG}^r(\mathcal{H})) = \{\{e_1, \ldots, e_r\} \ : \ e_1, \ldots, e_r \text{ are pairwise vertex-disjoint hyperedges of } H\}.$$

The Kneser hypergraph $\mathrm{KG}^r(n, k)$ is an r-uniform hypergraph obtained in this way when $\mathcal{H} = ([n], \binom{[n]}{k})$, i.e., the complete k-uniform hypergraph with $[n]$ as the vertex set. For $r = 2$, throughout the paper, we would rather use $\mathrm{KG}(\mathcal{H})$ than $\mathrm{KG}^2(\mathcal{H})$. It is a known simple fact that for any graph G, there are several hypergraphs $\mathcal{H}$ such that $\mathrm{KG}(\mathcal{H})$ and G are isomorphic. For $n \geq 2k$, if we set $\mathcal{H}_1 = ([n], \binom{[n]}{k})$ and $\mathcal{H}_2 = ([n], \binom{[n]}{k}_2)$, then $\mathrm{KG}(\mathcal{H}_1)$ and $\mathrm{KG}(\mathcal{H}_2)$ are respectively called *the Kneser graph* $\mathrm{KG}(n, k)$ and *the Schrijver graph* $\mathrm{SG}(n, k)$.

In a breakthrough paper, in 1978, Lovász [16] using algebraic topological tools proved that $\chi(\mathrm{KG}(n, k)) = n - 2k + 2$. Later, Schrijver [18] improved this result by proving that the Schrijver graphs are vertex critical subgraphs of Kneser graphs with the same chromatic number. Also, it was proved by Alon, Frankl, and Lovász [2] that

$$\chi(\mathrm{KG}^r(n, k)) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

This result gave an affirmative answer to a conjecture posed by Erdős [6].

For a graph G and a family of graphs $\mathcal{F}$, *the general Kneser graph* $\mathrm{KG}(G, \mathcal{F})$ has the set of all subgraphs of G isomorphic to some members of $\mathcal{F}$ as the vertex set and two vertices are adjacent if their corresponding vertices are edge-disjoint. This definition can be easily generalized to hypergraphs as follows. Let $\mathcal{H}$ be a hypergraph and $\mathcal{L}$ be a family of hypergraphs. Define *the general Kneser hypergraph* $\mathrm{KG}^r(\mathcal{H}, \mathcal{L})$ to be an r-uniform hypergraph with the vertex set consisting of all subhypergraphs of $\mathcal{H}$ isomorphic to some members of $\mathcal{L}$ and with the hyperedge set consisting of all r-tuples of vertices whose corresponding subhypergraphs are pairwise hyperedge-disjoint.

Remark 1. Let $\mathcal{K} = (V(\mathcal{K}), E(\mathcal{K}))$ be a subhypergraph of $\mathcal{H}$. Clearly, the set $V(\mathcal{K}) \setminus \bigcup_{T \in E(\mathcal{K})} T$ consists of all *isolated vertices* of $\mathcal{K}$. Therefore, if we set $V' = \bigcup_{T \in E(\mathcal{K})} T$, $E' = E(\mathcal{K})$, and $\mathcal{K}' = (V', E')$, then one can see that the general Kneser hypergraphs $\mathrm{KG}^r(\mathcal{H}, \mathcal{K})$ and $\mathrm{KG}^r(\mathcal{H}, \mathcal{K}')$ are homomorphic, that is, there is a homomorphism from $\mathrm{KG}^r(\mathcal{H}, \mathcal{K})$ to $\mathrm{KG}^r(\mathcal{H}, \mathcal{K}')$ and vice versa, which results in $\chi(\mathrm{KG}^r(\mathcal{H}, \mathcal{K})) = \chi(\mathrm{KG}^r(\mathcal{H}, \mathcal{K}'))$. In view of this observation, throughout the paper, for any general Kneser graph $\mathrm{KG}(\mathcal{H}, \mathcal{L})$, we assume that any $\mathcal{K} \in \mathcal{L}$ has no isolated vertex.

In view of the preceding remark, the general Kneser hypergraphs $\mathrm{KG}^r(\mathcal{H}, \mathcal{K})$ can be redefined as follows. If we define $\binom{\mathcal{H}}{\mathcal{L}}$ to be a hypergraph with the vertex set $E(\mathcal{H})$ and the hyperedge set

$$\{E(\mathcal{K}) \ : \ \mathcal{K} \text{ is a subhypergraph of } \mathcal{H} \text{ isomorphic to some member of } \mathcal{L}\},$$

then $\mathrm{KG}^r(\mathcal{H}, \mathcal{L}) = \mathrm{KG}^r(\binom{\mathcal{H}}{\mathcal{L}})$. Throughout the paper, when $\mathcal{L} = \{\mathcal{H}'\}$, we prefer to use $\mathrm{KG}^r(\mathcal{H}, \mathcal{H}')$ instead of $\mathrm{KG}^r(\mathcal{H}, \mathcal{L})$. Also, for $r = 2$, we would rather use $\mathrm{KG}(\mathcal{H}, \mathcal{L})$ than $\mathrm{KG}^2(\mathcal{H}, \mathcal{L})$. Moreover, $\mathrm{KG}(\mathcal{H}, \mathcal{L})$ will be called general Kneser graph.

In the following, for clarifying the definitions, some well-known families of graphs will be presented in terms of general Kneser graphs.

1. The Kneser graph $\mathrm{KG}(n, k)$ $(n \geq 2k)$ is isomorphic to $\mathrm{KG}(nK_2, kK_2)$.
2. The Schrijver graph $\mathrm{SG}(n, k)$ $(n \geq 2k)$ is isomorphic to $\mathrm{KG}(C_n, kK_2)$.