Discrete Mathematics 340 (2017) 2388–2397

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Sums of powers of Catalan triangle numbers

Pedro J. Miana [a,*], Hideyuki Ohtsuka [b], Natalia Romero [c]

[a] Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain
[b] Bunkyo University High School, 1191-7, Kami, Ageo-city, Saitama Pref., 362-0001, Japan
[c] Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain

ARTICLE INFO

Article history:
Received 21 August 2016
Received in revised form 16 April 2017
Accepted 13 May 2017

Keywords:
Catalan numbers
Combinatorial identities
Binomial coefficients
Catalan triangle

ABSTRACT

In this paper, we consider combinatorial numbers $(C_{m,k})_{m\geq 1,k\geq 0}$, mentioned as Catalan triangle numbers where $C_{m,k} := \binom{m-1}{k} - \binom{m-1}{k-1}$. These numbers unify the entries of the Catalan triangles $B_{n,k}$ and $A_{n,k}$ for appropriate values of parameters m and k, i.e., $B_{n,k} = C_{2n,n-k}$ and $A_{n,k} = C_{2n+1,n+1-k}$. In fact, these numbers are suitable rearrangements of the known ballot numbers and some of these numbers are the well-known Catalan numbers C_n that is $C_{2n,n-1} = C_{2n+1,n} = C_n$.

We present identities for sums (and alternating sums) of $C_{m,k}$, squares and cubes of $C_{m,k}$ and, consequently, for $B_{n,k}$ and $A_{n,k}$. In particular, one of these identities solves an open problem posed in Gutiérrez et al. (2008). We also give some identities between $(C_{m,k})_{m\geq 1,k\geq 0}$ and harmonic numbers $(H_n)_{n\geq 1}$. Finally, in the last section, new open problems and identities involving $(C_n)_{n\geq 0}$ are conjectured.

0. Introduction

The well-known Catalan numbers $(C_n)_{n\geq 0}$ given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0,$$

appear in a wide range of problems. For instance, the Catalan number C_n counts the number of ways to triangulate a regular polygon with $n+2$ sides; or, the number of ways that $2n$ people seat around a circular table are simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other, see for example [19,22].

The Catalan numbers may be defined recursively by $C_0 = 1$ and $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$ for $n \geq 1$ and first terms in this sequence are

$$1, \ 1, \ 2, \ 5, \ 14, \ 42, \ 132, \ldots.$$

Catalan numbers have been studied in depth in many papers and monographs (see for example [4–12,16–22]), and the Catalan sequence is probably the most frequently encountered sequence.

In this paper, we consider combinatorial numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ given by

$$C_{m,k} := \frac{m-2k}{m}\binom{m}{k} = \binom{m-1}{k} - \binom{m-1}{k-1}. \tag{0.1}$$

* Corresponding author.
E-mail addresses: pjmiana@unizar.es (P.J. Miana), otsukahideyuki@gmail.com (H. Ohtsuka), natalia.romero@unirioja.es (N. Romero).

http://dx.doi.org/10.1016/j.disc.2017.05.006

Note that the number $C_{m,k}$ gives the difference of ways to choose a subset of size k instead of $k-1$ elements, disregarding their order, from a set of m elements. We collect the first values in the following table:

$m \setminus k$	0	1	2	3	4	5	6	7	8	9	10	…
1	1	−1										
2	1	0	−1									
3	1	1	−1	−1								
4	1	2	0	−2	−1							
5	1	3	2	−2	−3	−1						
6	1	4	5	0	−5	−4	−1					
7	1	5	9	5	−5	−9	−5	−1				
8	1	6	14	14	0	−14	−14	−6	−1			
9	1	7	20	28	14	−14	−28	−20	−7	−1		
10	1	8	27	48	42	0	−42	−48	−27	−8	−1	
…	…	…	…	…	…	…	…	…	…	…	…	…

(0.2)

Notice that, these combinatorial numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ are suitable rearrangements of the known ballot numbers $(a_{m,k})$ with $a_{m,k}=\frac{k+1}{m+1}\binom{2m-k}{m}$ for $m\geq 0$ and $0\leq k\leq m$, i.e.,

$$a_{m,k}=C_{2m+1-k,m-k}, \qquad C_{m,k}=a_{m-k-1,m-2k-1},$$

see example [1]. Moreover, although these numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ have been not systematically treated in the literature, some identities may be rewritten in terms of them, for example, the following identity

$$\sum_{k=0}^{n} C_{m,k}\binom{m}{k}^2=\binom{m-1}{n}\sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1}, \tag{0.3}$$

was proven in [13] for $m, n\geq 1$.

These combinatorial numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ are closely related to Catalan numbers $(C_n)_{n\geq 0}$ and appear in several Catalan triangles. For instance, $C_{2n,n-k}=B_{n,k}$, where

$$B_{n,k}=\frac{k}{n}\binom{2n}{n-k}, \quad 0\leq k\leq n,$$

(see [16]) and also $C_{2n+1,n+1-k}=A_{n,k}$, where

$$A_{n,k}=\frac{2k-1}{2n+1}\binom{2n+1}{n+1-k}, \quad 1\leq k\leq n+1,$$

(see [12]). The sequence $(A_{n,k})$ is an example of Catalan-like numbers considered in [2].

This paper is organized as follows. In the first section, we establish the sum of $C_{m,k}$ and their alternating sums, $(-1)^k C_{m,k}$ in Theorem 1.2. Next, as a consequence in Corollary 1.3, we obtain the alternating sum of the entries of the two Catalan triangle numbers $(B_{n,k})_{n\geq k\geq 1}$ and $(A_{n,k})_{n+1\geq k\geq 1}$. Also, we present a recurrence relation which is satisfied by the numbers $(C_{m,k})_{m\geq 1,k\geq 0}$.

Identities which involved harmonic numbers $(H_n)_{n\geq 1}$ where

$$H_n=\sum_{k=1}^{n}\frac{1}{k}, \quad n\in\mathbb{N}, \tag{0.4}$$

have received a notable attention in last decades. We only mention shortly papers [5,14,20], the monograph [3, Chapter 7] and the reference therein.

At the end of the first section, we present a new identity which involves harmonic numbers $(H_n)_{n\geq 1}$ and Catalan triangle numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ in Theorem 1.4 (and then for $B_{n,k}$ and $A_{n,k}$ in Corollary 1.5). This identity includes, as particular case, a known equality proved in [14].

In the second section, we obtain the value of $\sum_{k=0}^{n} C_{m,k}^2$ and $\sum_{k=0}^{n}(-1)^k C_{m,k}^2$ for $m, n\geq 1$ in Theorem 2.1. We also show two identities which allow us to decompose squares of combinatorial numbers as sum of squares of other combinatorial numbers. In particular, the nice identity

$$\binom{2n}{n}^2=\sum_{k=0}^{n}\frac{3n-2k}{n}\binom{2n-1-k}{n-1}^2, \qquad n\geq 1,$$

is presented in Theorem 2.3 and seems to be new.

The third section is dedicated to the sum of cubes of numbers $(C_{m,k})_{m\geq 1,k\geq 0}$. For $m\geq 1$ and $n\geq 1$, we present the identity

$$\sum_{k=0}^{n} C_{m,k}^3=4\binom{m-1}{n}^3-3\binom{m-1}{n}\sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1},$$