Discrete Mathematics 340 (2017) 2402–2407

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Distant sum distinguishing index of graphs

Jakub Przybyło [1]

AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland

ARTICLE INFO

Article history:
Received 16 December 2016
Received in revised form 19 May 2017
Accepted 19 May 2017

Keywords:
Distant sum distinguishing index of a graph
Neighbour sum distinguishing index
Adjacent strong chromatic index
Distant set distinguishing index

ABSTRACT

Consider a positive integer r and a graph $G = (V, E)$ with maximum degree Δ and without isolated edges. The least k so that a proper edge colouring $c : E \to \{1, 2, \ldots, k\}$ exists such that $\sum_{e\ni u} c(e) \neq \sum_{e\ni v} c(e)$ for every pair of distinct vertices u, v at distance at most r in G is denoted by $\chi'_{\Sigma,r}(G)$. For $r = 1$, it has been proved that $\chi'_{\Sigma,1}(G) = (1+o(1))\Delta$. For any $r \geq 2$ in turn an infinite family of graphs is known with $\chi'_{\Sigma,r}(G) = \Omega(\Delta^{r-1})$. We prove that, on the other hand, $\chi'_{\Sigma,r}(G) = O(\Delta^{r-1})$ for $r \geq 2$. In particular, we show that $\chi'_{\Sigma,r}(G) \leq 6\Delta^{r-1}$ if $r \geq 4$.

1. Introduction

Vertex distinguishing edge colourings have their origins in the concept of *irregularity strength*. This graph invariant was designed in [10] as a peculiar measure of a "level of irregularity" of a graph. A graph or multigraph is called *irregular* itself if all its vertices have pairwise distinct degrees (see [9] for possible alternative definitions). Note that there are in fact no irregular graphs at all, except the trivial 1 vertex case. Thus, to capture the degree of irregularity of a graph, the authors of [10] exploited the fact that there are in turn irregular multigraphs of any order, except order 2. The irregularity strength of a graph $G = (V, E)$, $s(G)$, is then defined as the least k such that we are able to construct an irregular multigraph of a given graph by multiplying some of its edges — each at most k times. Equivalently, it is the least k so that an edge colouring $c : E \to \{1, 2, \ldots, k\}$ exists attributing every vertex $v \in V$ a distinct *weighted degree* defined as follows:

$$d_c(v) := \sum_{e\ni v} c(e).$$

This shall be also called the *sum at* v, see e.g. [3,6,11,12,14,15,17,24,27,28,30,33,34] for exemplary results concerning $s(G)$. An intriguing local version of the same problem was proposed in [25]. The parameter investigated there differs from $s(G)$ by the reduction of the pairwise distinction requirement only to *adjacent* vertices, and shall be denoted by $s_1(G)$. The well known *1–2–3 Conjecture* presumes that $s_1(G) \leq 3$ for every graph G without isolated edges, see [25]. This was investigated e.g. in [1,2,41]. In general, it is however thus far only known that $s_1(G) \leq 5$, see [23]. A distance generalization of this problem, introduced in [35] and referring in particular to the known distant chromatic numbers (see [26] for a survey of this topic), handles a graph invariant $s_r(G)$ (where r is a positive integer), that is the least integer k so that an edge colouring $c : E \to \{1, 2, \ldots, k\}$ exists with $d_c(u) \neq d_c(v)$ for every $u, v \in V$ at distance at most r in G, $u \neq v$ — see also [31].

The main subject of this paper is the correspondent of $s_r(v)$ in the case of *proper* edge colourings. For any positive integer r and a graph $G = (V, E)$ without isolated edges, by $\chi'_{\Sigma,r}(G)$ we denote the least integer k such that a proper edge colouring $c : E \to \{1, 2, \ldots, k\}$ exists with $d_c(u) \neq d_c(v)$ for every $u, v \in V$ with $1 \leq d(u, v) \leq r$, where $d(u, v)$ denotes the distance of

E-mail address: jakubprz@agh.edu.pl.

[1] Fax: +048-12-617-31-65.

http://dx.doi.org/10.1016/j.disc.2017.05.009

u and v in G. This is called the *r-distant sum distinguishing index* of G. Such concept develops the study on the earlier *neighbour sum distinguishing index* of G, $\chi'_\Sigma(G) = \chi'_{\Sigma,1}(G)$, for which it was conjectured in [16] that $\chi'_\Sigma(G) \leq \Delta(G)+2$ for any connected graph G of order at least three different from the cycle C_5. This was asymptotically confirmed in [37] and [39], where it was showed that $\chi'_\Sigma(G) \leq (1+o(1))\Delta(G)$, see also [8,13,16,36,38,40] for other results concerning χ'_Σ.

Exactly the same upper bound as in the case of χ'_Σ above was conjectured to hold for the graph invariant $\chi'_a(G)$ [43] (so called *adjacent strong chromatic index* of G), i.e. the least k for which a proper edge colouring $c : E \to \{1, 2, \dots, k\}$ exists attributing distinct sets of incident colours to the neighbours in G (see e.g. [4,5,18–22,42,43] for a number of partial results and upper bounds for this graph invariant, which is one of the most intensively studied subjects within the area), though obviously $\chi'_a(G) \leq \chi'_\Sigma(G)$ for every graph G without isolated edges. It is however much more challenging to distinguish vertices by sums than by the corresponding sets (even though the conjectured optimal upper bounds are the same in case of the both parameters — χ'_Σ and χ'_a), what can be easily seen while attempting to apply the probabilistic method. Such approach was e.g. used in [19] to provide an upper bound $\chi'_a(G) \leq \Delta(G)+C$ for all graphs without isolated edges where C is a constant (in particular, if $\Delta(G)$ is large enough, $C = 300$ suffices). In order to bring out the fact that distinguishing by sums is indeed much more demanding than by sets, one needs to consider distance correspondents of χ'_Σ and χ'_a. It was in particular conjectured in [32] that for any $r \geq 2$, analogously as in the case of $r = 1$, $\chi'_{a,r}(G) \leq \Delta(G) + C$ under minor assumption that $\delta(G) \geq \delta_0$, where C and δ_0 are constants dependent on r. This was confirmed asymptotically and also exactly for some wide graph classes, in particular for all regular (and almost regular) graphs with degree large enough, see [32] for details. The same certainly does not hold in case of distinguishing by sums, though. Indeed, from [35] follow lower bounds for $\chi'_{\Sigma,r}$ based on research concerning so-called Moore bound (see e.g. a survey [29] concerning this), focused on studying the largest possible number of vertices of a graph with maximum degree Δ and diameter r, denoted by $n_{\Delta,r}$. Namely, it is known that $\chi'_{\Sigma,r}(G) \geq s_r(G) \geq \frac{n_{\Delta,r}}{\Delta}$; hence, using e.g. a construction of *undirected de Bruijn graphs*, we get for every $r \geq 2$ an infinite family of graph with $\chi'_{\Sigma,r}(G) \geq \Omega(\Delta^{r-1})$, while using an asymptotic result of Bollobás and Fernandez de la Vega [7] we even obtain for a fixed Δ an infinite family of graphs with diameter r tending to infinity of order asymptotically equivalent to Δ^r (hence with $\chi'_{\Sigma,r}(G)$ at least asymptotically equivalent to Δ^{r-1}), see [35] for details. Lower bounds of the same form also hold if we narrow our interest down to regular (or almost regular) graphs. This shows that the difference between the behaviour of $\chi'_{\Sigma,r}$ and $\chi'_{a,r}$ is enormous for $r > 2$, what could not be discerned e.g. in case of distinguishing only neighbours (i.e. for $\chi'_\Sigma = \chi'_{\Sigma,1}$ and $\chi'_a = \chi'_{a,1}$).

In this paper, we provide general upper bounds for $\chi'_{\Sigma,r}$ of the same magnitude as the lower ones above. In particular, we prove that $\chi'_{\Sigma,r}(G) \leq 6\Delta^{r-1}$ for $r \geq 4$ and prove the upper bound of order Δ^{r-1} also in the remaining cases (for $r = 2, 3$), see Theorem 1 for details. These are the first upper bounds of this order for these graph invariants, refining the result from [35], where only slightly better upper bounds (with the same leading ingredient) were proved to hold for the simpler case of non-proper edge colourings, i.e., the graph invariants $s_r(G)$.

2. Main result and proof

The above-mentioned Moore bound, expressing an upper bound for the largest possible number of vertices of a graph with maximum degree Δ and diameter r, is the following (see [29]):

$$M_{\Delta,r} := 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{r-1}.$$

Given a graph $G = (V, E)$ with maximum degree Δ and a vertex $v \in V$, denote by $N^r(v)$ the set of *r-neighbours* of v, i.e. vertices $u \neq v$ at distance at most r from v in G, and note that $d^r(v) := |N^r(v)| \leq M_{\Delta,r} - 1 \leq \Delta^r$ for any $r \geq 1$.

Theorem 1. *Let G be a graph without isolated edges and with maximum degree $\Delta \geq 2$, and let r be an integer, $r \geq 2$. Then,*

$$\chi'_{\Sigma,r}(G) \leq 6\left(\frac{M_{\Delta,r}-1}{\Delta} + \Delta - 1\right) + \Delta,$$

hence

$$\chi'_{\Sigma,r}(G) \leq 6\Delta^{r-1}$$

for $r \geq 4$, while $\chi'_{\Sigma,3}(G) \leq 6\Delta^2 + \Delta$ and $\chi'_{\Sigma,2}(G) \leq 13\Delta - 6$.

Proof. We fix $r \geq 2$ and prove the theorem by induction with respect to the number of vertices of G, denoted by n. It is sufficient to show the thesis in the case when G is a connected graph (which is not an isolated edge) with maximum degree $\Delta \geq 2$.

For $n = 3$, the theorem obviously holds, so assume $n \geq 4$. Denote

$$M = \frac{M_{\Delta,r}-1}{\Delta} \quad \text{and} \quad K = M + \Delta - 1,$$

and note that then for every $v \in V$ we in particular have

$$d^r(v) \leq d(v)M. \tag{1}$$