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Graphs and their associated inverse semigroups

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ABSTRACT

Directed graphs have long been used to gain an understanding of the structure of semigroups, and recently the structure of directed graph semigroups has been investigated resulting in a characterization theorem and an analog of Frucht's Theorem. We investigate two inverse semigroups defined over undirected graphs constructed from the notions of subgraph and vertex set induced subgraph. We characterize the structure of the semilattice of idempotents and lattice of ideals of these inverse semigroups. We prove a characterization theorem that states that every graph has a unique associated inverse semigroup up to isomorphism allowing for an algebraic restatement of the Edge Reconstruction Conjecture.

1. Introduction

Much of the theory linking semigroups to graphs have been in the guise of directed graphs [2,8,10,11,15,18]. However, undirected graphs have rich internal symmetries for which groups are too coarse an algebraic structure to distinguish. This has lead to the notions of distinguishing number [1] and fixing number [6]. Furthermore, local symmetry in the form of subgraph embeddings has been used famously by Lovász to solve the edge reconstruction conjecture [7] for graphs with n vertices and m edges where $m \geq 1/2\binom{n}{2}$ [14]. A possible algebraic structure to study local symmetry is an inverse semigroup. This leads us to investigate inverse semigroups on undirected graphs.

In Section 2, we begin by defining inverse semigroups associated to undirected graphs to correspond to the ideas of subgraph symmetry and vertex set induced subgraph symmetry. These two inverse semigroups are linked to the edge reconstruction conjecture [7] and the vertex reconstruction conjecture [3]. These inverse semigroups are graph analogs of the inverse semigroup of sets [16] with a necessary restriction of partial monomorphism to partial isomorphism [5].

In Section 3, we characterize the idempotents and ideals for both inverse semigroups. In Section 4, we prove that the inverse semigroup corresponding to all subgraphs of a graph uniquely determines that graph up to isomorphism. Finally, in Section 5, we provide an algebraic restatement of the Edge Reconstruction Conjecture using inverse semigroups and consider the difficulties faced when attempting to form a similar restatement for the Vertex Reconstruction Conjecture.

We will follow the notations of [4] for graph theory, and [9] for inverse semigroups. Specifically, given a graph G we denote the incidence function of G by ψ_G. We will only consider finite undirected graphs, but they are allowed to have multiple edges and loops. Given two graphs G and H, we define a *graph isomorphism* $\varphi : G \to H$ as a pair of bijections, $\varphi_V : V(G) \to V(H)$ and $\varphi_E : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\varphi_E(e)) = \varphi_V(u)\varphi_V(v)$ for all vertices u and v of G. We allow $\emptyset$ to be considered a graph without vertices or edges and $\mu_0 : \emptyset \to \emptyset$ to be a valid graph isomorphism, where the bijections between the empty vertex sets and empty edge sets are empty maps.

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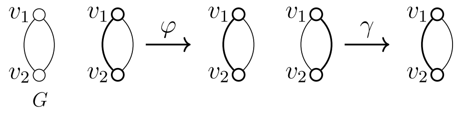

Fig. 1. A graph and two subgraph isomorphisms for $G = (\{v_1, v_2\}, \{e_1, e_2\})$.

2. Inverse semigroups constructed from graph symmetry

We will start with the most general inverse semigroup associated to all subgraphs of a graph G. This is an inverse semigroup defined on the set of all partial isomorphisms of the subgraphs of G, including the empty graph, where the operation is a modified composition.

Example 2.1. For an example of this operation, consider the graph G in Fig. 1 consisting of vertex set $\{v_1, v_2\}$ with multiple edges e_1 and e_2 both incident to v_1 and v_2. Define subgraph isomorphism $\varphi : (\{v_1, v_2\}, \{e_1\}) \to (\{v_1, v_2\}, \{e_1\})$ by setting both φ_V and φ_E to be identity functions. Define subgraph isomorphism $\gamma : (\{v_1, v_2\}, \{e_2\}) \to (\{v_1, v_2\}, \{e_1\})$ by setting γ_V to be the identity function and setting $\gamma_E(e_2) = e_1$. Notice that the intersection of image of φ_E and the domain of γ_E is empty. Hence, $\gamma\varphi$ is the identity subgraph isomorphism from $(\{v_1, v_2\}, \emptyset)$ to itself. However, as the intersection of the image of γ_E and the domain of φ_E is $\{e_1\}$, $\varphi\gamma$ is the subgraph isomorphism from $(\{v_1, v_2\}, \{e_2\})$ to $(\{v_1, v_2\}, \{e_1\})$ where $\varphi\gamma_V$ is the identity function and $\varphi\gamma_E(e_2) = e_1$.

Definition 2.2. Let G be a graph. We define the *full inverse semigroup of* G, denoted $\mathrm{Fisg}(G)$, to be the collection of all graph isomorphisms $\varphi : H \to J$ where H and J are subgraphs of G. We define the operation to be composition of isomorphisms.

We now define the operation, denoted by multiplication, for this semigroup. For $\gamma, \varphi \in \mathrm{Fisg}(G)$ we define $\gamma\varphi : \varphi^{-1}(\mathrm{Dom}(\gamma)) \to \gamma(\mathrm{Im}(\varphi))$ to be $\gamma \circ \varphi|_{\varphi^{-1}(\mathrm{Dom}(\gamma))}$, where $\varphi|_{\varphi^{-1}(\mathrm{Dom}(\gamma))} : \varphi^{-1}(\mathrm{Dom}(\gamma)) \to \mathrm{Dom}(\gamma)$ is the restriction graph isomorphism with vertex map $\varphi_V|_{\varphi^{-1}(\mathrm{Dom}(\gamma_V))} : V(\varphi^{-1}(\mathrm{Dom}(\gamma))) \to V(\mathrm{Dom}(\gamma))$ and edge map $\varphi_E|_{\varphi^{-1}(\mathrm{Dom}(\gamma_E))} : E(\varphi^{-1}(\mathrm{Dom}(\gamma))) \to E(\mathrm{Dom}(\gamma))$. In the case that the image of φ and the domain of γ are disjoint, $\gamma\varphi$ will be μ_0, the identity isomorphism of the empty set graph. It is easy to see that $\gamma\varphi$ is an isomorphism of subgraphs, and the operation is well defined.

The operation of $\mathrm{Fisg}(G)$ is associative, and for any subgraph isomorphism φ, $\varphi\varphi^{-1}\varphi = \varphi$. Hence, $\mathrm{Fisg}(G)$ forms an inverse semigroup under the operation defined above. As we will see in Section 5, $\mathrm{Fisg}(G)$ is directly connected to the edge reconstruction conjecture.

We obtain an analogous connection for the vertex reconstruction conjecture if we instead consider an inverse semigroup associated to subgraphs of G induced by vertex sets.

Definition 2.3. Let G be a graph. We define the *induced inverse semigroup* of G, denoted $\mathrm{Iisg}(G)$, to be the collection of all graph isomorphisms $\varphi : H \to J$ where H and J are vertex subset induced subgraphs of G. We define the operation to be composition of isomorphisms.

We then define the semigroup operation in the same way as for $\mathrm{Fisg}(G)$. Note that the intersection of two vertex induced subgraphs is a vertex induced subgraph when the empty vertex set and edge set graph is considered as the induced subgraph on $\emptyset$. Hence, $\mathrm{Iisg}(G)$ is also an inverse semigroup under the operation inherited from $\mathrm{Fisg}(G)$.

3. Idempotent and ideal structure

In this section, we concern ourselves with the idempotents and ideals of our inverse semigroups. We categorize the idempotents and ideals of our inverse semigroups.

Lemma 3.1. *If G is a graph, then $e \in \mathrm{Fisg}(G)$ is idempotent if and only if there exists a subgraph H of G with $e = \mathrm{id}_H$, the identity automorphism.*

Proof. ($\Leftarrow$) For every subgraph H, id_H is idempotent.

($\Rightarrow$) Let $e \in \mathrm{Fisg}(G)$ be an idempotent. As $ee = e$, $\mathrm{Dom}(e) = \mathrm{Im}(e)$ and e is an automorphism of a subgraph $H = \mathrm{Dom}(e)$. As $\mathrm{Aut}(H)$ is a group, and the only idempotent of a group is the identity, $e = \mathrm{id}_H$. $\square$

For any graph G, $\mathrm{Iisg}(G)$ is a subsemigroup of $\mathrm{Fisg}(G)$. Thus, the idempotents are the identities of vertex induced subgraphs of G. We then note that the semilattice of idempotents of $\mathrm{Fisg}(G)$ is isomorphic to the semilattice of subgraphs of G, and similarly the semilattice of idempotents of $\mathrm{Iisg}(G)$ is isomorphic to the semilattice of induced subgraphs of G (see Figs. 2 and 3).

We now discuss *ideals* of a semigroup. Given a semigroup S and $I \subseteq S$, we say I is an ideal of S if $IS, SI \subseteq I$. This is of particular importance since ideals of a semigroup induce an equivalence relation which leads to the construction of a quotient