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The perimeter of words

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ABSTRACT

We define $[k] = \{1, 2, \ldots, k\}$ to be a (totally ordered) *alphabet* on k letters. A *word* w of length n on the alphabet $[k]$ is an element of $[k]^n$. A word can be represented by a bargraph (i.e., by a column-convex polyomino whose lower edges lie on the x-axis) in which the height of the ith column equals the size of the ith part of the word. Thus these bargraphs have heights which are less than or equal to k. We consider the perimeter, which is the number of edges on the boundary of the bargraph. By way of Cramer's method and the kernel method, we obtain the generating function that counts the perimeter of words. Using these generating functions we find the average perimeter of words of length n over the alphabet $[k]$. We also show how the mean and variance can be obtained using a direct counting method.

1. Introduction

Establishing a generating function for polyominoes according to perimeter is important to chemists, physicists and mathematicians alike (see [1,3,8,10,13]). Counting **all** polyominoes in this way is currently beyond reach of mathematicians and has been done only for some special subclasses of polyominoes as sited above. Here, we focus on extending this knowledge to the class of height restricted bargraphs (i.e., words over an alphabet $[k]$).

We define $[k] = \{1, 2, 3, \ldots, k\}$ to be a (totally ordered) *alphabet* on k letters. A *word* w of length n on the alphabet $[k]$ is an element of $[k]^n$. Each letter is also called a part. Various properties of words have been studied in [12] and [14–16]. A word can be represented by a bargraph (i.e., a column-convex polyomino whose lower edges lie on the x-axis).

These bargraphs are drawn on a regular planar lattice grid and are made up of square cells. A word over alphabet $[k]$ is uniquely defined by the size of each part. The height of the ith column of the representing bargraph matches the size of the ith part. The length of the word is the number of columns in the representing bargraph; each column with height at most k.

The perimeter is the number of edges on the boundary of the bargraph (see Fig. 1). The enumeration of polyominoes according to their area and perimeter has been well documented (see for example [1,3–11,13]). In this paper we extend these studies to the perimeter of words over the alphabet $[k]$, where the words are represented by their associated bargraphs. The perimeter of these objects has as far as we know not been studied before in the literature. The enumeration of unrestricted bargraphs according to several other parameters has been studied in [2].

We illustrate, in Fig. 1, the perimeter of the word 25463 over any alphabet $[k]$ where $k \geq 6$. The perimeter here is 24.

From now on, we will consider a word to be its associated bargraph of height at most k.

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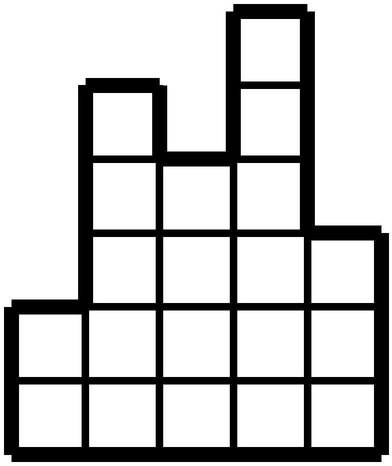

Fig. 1. Perimeter of 24 for the word 25463.

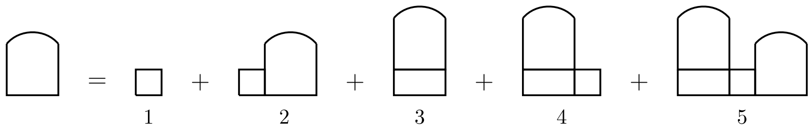

Fig. 2. Wasp-waist factorisation of bargraphs.

2. The generating function for the perimeter of words

In this section we obtain generating functions that count the perimeter of words over an alphabet k using 3 different methods: The wasp-waist method, the scanning-element algorithm and a method that uses linear algebra.

2.1. Wasp-waist method

Let $P_k(x, p)$ be the generating function for words over alphabet k, where x counts the number of columns and p the perimeter. We apply the wasp-waist decomposition (see [2]) to words over the alphabet $[k + 1]$ (see Fig. 2):
we have

$$P_{k+1}(x, p) = xp^4 + xp^2P_{k+1} + p^2P_k + xp^4P_k + xp^2P_kP_{k+1}.$$

Hence

$$P_{k+1} = \frac{xp^4 + p^2(1 + xp^2)P_k}{1 - xp^2 - xp^2P_k}.$$

Since the P_k's are rational functions we can write P_k as $\frac{s_k}{t_k}$, then we have

$$\frac{s_{k+1}}{t_{k+1}} = \frac{xp^4 + p^2(1 + p^2x)\frac{s_k}{t_k}}{1 - xp^2 - xp^2\frac{s_k}{t_k}} = \frac{xp^4t_k + p^2(1 + p^2x)s_k}{(1 - xp^2)t_k - xp^2s_k}.$$

Thus

$$s_{k+1} = xp^4t_k + p^2(1 + p^2x)s_k, \qquad \text{and} \qquad t_{k+1} = (1 - xp^2)t_k - xp^2s_k.$$

We write this in matrix form $\binom{s_{k+1}}{t_{k+1}} = A\binom{s_k}{t_k}$ where

$$A = \begin{pmatrix} p^2(1 + xp^2) & xp^4 \\ -xp^2 & 1 - xp^2 \end{pmatrix}.$$

After iterating k times we obtain $\binom{s_{k+1}}{t_{k+1}} = A^k\binom{s_1}{t_1}$, with initial conditions

$$\frac{s_1}{t_1} = P_1 = xp^4 + x^2p^6 + x^3p^8 + \cdots = \frac{xp^4}{1 - xp^2}$$