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Toll number of the Cartesian and the lexicographic product of graphs

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ABSTRACT

Toll convexity is a variation of the so-called interval convexity. A tolled walk T between two non-adjacent vertices u and v in a graph G is a walk, in which u is adjacent only to the second vertex of T and v is adjacent only to the second-to-last vertex of T. A toll interval between $u, v \in V(G)$ is a set $T_G(u, v) = \{x \in V(G) : x \text{ lies on a tolled walk between } u \text{ and } v\}$. A set $S \subseteq V(G)$ is toll convex, if $T_G(u, v) \subseteq S$ for all $u, v \in S$. A toll closure of a set $S \subseteq V(G)$ is the union of toll intervals between all pairs of vertices from S. The size of a smallest set S whose toll closure is the whole vertex set is called a toll number of a graph G, $tn(G)$. The first part of the paper reinvestigates the characterization of convex sets in the Cartesian product of two graphs. It is proved that the toll number of the Cartesian product of two graphs equals 2. In the second part, the toll number of the lexicographic product of two graphs is studied. It is shown that if H is not isomorphic to a complete graph, $tn(G \circ H) \leq 3 \cdot tn(G)$. We give some necessary and sufficient conditions for $tn(G \circ H) = 3 \cdot tn(G)$. Moreover, if G has at least two extreme vertices, a complete characterization is given. Furthermore, graphs with $tn(G \circ H) = 2$ are characterized. Finally, the formula for $tn(G \circ H)$ is given — it is described in terms of the so-called toll-dominating triples or, if H is complete, toll-dominating pairs.

1. Introduction

Theory of convex structures developed from the classical convexity in Euclidean spaces and resulted in the *abstract convexity* theory. It is based on three natural conditions, imposed on a family of subsets of a given set. All three axioms hold in the so-called *interval convexity*, which was emphasized in [19] as one of the most natural ways for introducing convexity. An interval $I : X \times X \to 2^X$ has the property that $x, y \in I(x, y)$, and convex sets are defined as the sets S in which all intervals between elements from S lie in S. In terms of graph theory, several interval structures have been introduced. The interval function I is most commonly defined by a set of paths between two vertices, where these paths have some interesting properties. For instance, shortest paths yield geodesic intervals, induced paths yield monophonic intervals etc. Each type of an interval then gives rise to the corresponding convexity, see [8,17] for some basic types of intervals/convexities.

Many properties were investigated in different interval convexities. One of the most natural ones arises from the abstract convexity theory and is called convex geometry property or Minkowski–Krein–Milman property. Recall that a vertex s from a convex set S is an *extreme vertex* of S, if $S - \{s\}$ is also convex. A graph G is called a *convex geometry* with respect to a given convexity, if any convex set of G is the convex hull of its extreme vertices. In the case of monophonic convexity, exactly chordal graphs are convex geometries, while in the geodesic convexity, these are precisely Ptolemaic graphs (i.e. distance-hereditary chordal graphs), see [13].

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A graph convexity, for which exactly the interval graphs are convex geometry, was investigated and introduced in [2]. Authors focused their attention on interval graphs and applied the concept from [1], where this family of graphs was characterized in terms of tolled walks. The definition of a toll walk, which is a generalization of monophonic paths, as any monophonic path is also a tolled walk, yields the definition of the toll convexity. In [2], authors considered other properties of the toll convexity followed by previously considered types of convexities. They focused on the toll number and the t-hull number of a graph, which were investigated in terms of the geodesic convexity about 30 years ago [12,15] and intensively studied after that, for instance in graph products [3,5–7], in terms of other types of convexities [11,18] and more, see [9,10,17] for further reading on this topic. In this paper we will study these two invariants on the Cartesian and the lexicographic product of two graphs.

We proceed as follows. In Section 2 we present main definitions and results from [2] that are needed all over the paper. Section 3 is devoted to explore the toll convexity on the Cartesian product of graphs. First, a counterexample to the characterization of t-convex sets in the Cartesian product from [2], in which the authors missed one condition, is presented. We fix this characterization and use the result to prove that the t-hull number of the Cartesian product of two connected, non-complete graphs equals 2. Then our study of the toll number on the Cartesian product of two graphs yields to the result that it also equals 2 for any two connected non-trivial graphs.

In the subsequent section we again use the result from [2] to show that the t-hull number of the lexicographic product of two connected non-trivial graphs G and H, where H is not complete, equals 2. A characterization of graphs with $tn(G\circ H) = 2$ is completely solved. Further on, some bounds for the toll number of the lexicographic product of two graphs and some necessary and sufficient conditions for $tn(G \circ H) = 3 \cdot tn(G)$ are given in case H is not a complete graph. If G has at least two extreme vertices (i.e. $|Ext(G)| \geq 2$), then a characterization of graphs with $tn(G\circ H) = 3\cdot tn(G)$ is shown. Finally we establish a formula that expresses the exact toll number of $G \circ H$ using new concepts, obtained from the same idea as geodominating triple in [5]. The paper is finished by the short section of open problems.

2. Preliminaries

All graphs, considered in this paper, are finite, simple, non-trivial (i.e. graphs with at least two vertices), connected and without multiple edges or loops.

Let $G = (V(G), E(G))$ be a graph. The *distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. The *diameter* of a graph, $diam(G)$, is defined as $diam(G) = \max_{u,v\in V(G)}\{d(u, v)\}$.

The *geodesic interval* $I_G(u, v)$ between vertices u and v is the set of all vertices that lie on some shortest path between u and v in G, i.e. $I_G(u, v) = \{x \in V(G) : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$. A subset S of $V(G)$ is *geodesically convex* (or *g-convex*) if $I_G(u, v) \subseteq S$ for all $u, v \in S$. Let S be a set of vertices of a graph G. Then the geodetic closure $I_G[S]$ is the union of geodesic intervals between all pairs of vertices from S, that is, $I_G[S] = \bigcup_{u,v\in S} I_G(u, v)$. A set S of vertices of G is a *geodetic set* in G if $I_G[S] = V(G)$. The size of a minimum geodetic set in a graph G is called the *geodetic number* of G and denoted by $g(G)$. Given a subset $S \subseteq V(G)$, the *convex hull* $[S]$ of S is the smallest convex set that contains S. We say that S is a *hull set* of G if $[S] = V(G)$. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$. Indices above may be omitted, whenever the graph G is clear from the context.

All definitions, listed above for the geodesic convexity, could be rewritten in terms of monophonic convexity, all-path convexity, Steiner convexity etc. For more details see surveys [4,10], the book [17] and the paper [16]. In the rest of the paper, the term convexity will always stand for the so-called toll convexity, unless we will say otherwise.

Let u and v be two different, non-adjacent vertices of a graph G. A *tolled walk* T between u and v in G is a sequence of vertices of the form

$$T : u, w_1, \ldots, w_k, v,$$

where $k \geq 1$, which enjoys the following three conditions:

- $w_iw_{i+1} \in E(G)$ for all i,
- $uw_i \in E(G)$ if and only if $i = 1$,
- $vw_i \in E(G)$ if and only if $i = k$.

In other words, a tolled walk is any walk between u and v such that u is adjacent only to the second vertex of the walk and v is adjacent only to the second-to-last vertex of the walk. For $uv \in E(G)$ let $T : u, v$ be a tolled walk as well. The only tolled walk that starts and ends in the same vertex v is v itself. We define $T_G(u, v) = \{x \in V(G) : x$ lies on a tolled walk between u and $v\}$ to be the *toll interval* between u and v in G. Finally, a subset S of $V(G)$ is *toll convex* (or *t-convex*) if $T_G(u, v) \subseteq S$ for all $u, v \in S$. The *toll closure* $T_G[S]$ of a subset $S \subseteq V(G)$ is the union of toll intervals between all pairs of vertices from S, i.e. $T_G[S] = \bigcup_{u,v\in S} T_G(u, v)$. If $T_G[S] = V(G)$, we call S a *toll set* of a graph G. The size of a minimum toll set in G is called the *toll number* of G and is denoted by $tn(G)$. Again, when graph is clear from the context, indices may be omitted.

A *t-convex hull* of a set $S \subseteq V(G)$ is defined as the intersection of all t-convex sets that contain S and is denoted by $[S]_t$. A set S is a *t-hull set* of G if its t-convex hull $[S]_t$ coincides with $V(G)$. The *t-hull number* of G is the size of a minimum t-hull set and is denoted by $th(G)$. Given the toll interval $T_G : V \times V \to 2^V$ and a set $S \subset V(G)$ we define $T_G^k[S]$ as follows: $T_G^0[S] = S$ and $T_G^{k+1}[S] = T_G[T_G^k[S]]$ for any $k \geq 1$. Note that $[S]_t = \bigcup_{k\in\mathbb{N}} T_G^k[S]$.