Discrete Mathematics 340 (2017) 2538–2549

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Injective choosability of subcubic planar graphs with girth 6

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ARTICLE INFO

Article history:
Received 10 November 2016
Received in revised form 2 May 2017
Accepted 31 May 2017

Keywords:
Planar graph
Injective coloring
Subcubic

ABSTRACT

An injective coloring of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor have distinct colors. A graph G is injectively k-choosable if for any list assignment L, where $|L(v)| \geq k$ for all $v \in V(G)$, G has an injective L-coloring. Injective colorings have applications in the theory of error-correcting codes and are closely related to other notions of colorability. In this paper, we show that subcubic planar graphs with girth at least 6 are injectively 5-choosable. This strengthens the result of Lužar, Škrekovski, and Tancer that subcubic planar graphs with girth at least 7 are injectively 5-colorable. Our result also improves several other results in particular cases.

1. Introduction

A *proper coloring* of a graph G is an assignment of colors to the vertices of G so that any neighboring vertices receive distinct colors. The *chromatic number* of G, $\chi(G)$, is the minimum number of colors needed for a proper coloring of G. An *injective coloring* of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. The *injective chromatic number*, $\chi_i(G)$, is the minimum number of colors needed for an injective coloring of G. An injective coloring of G is not necessarily a proper coloring of G. Define the *neighboring graph* $G^{(2)}$ by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv : u \text{ and } v \text{ have a common neighbor in } G\}$. Note that $\chi_i(G) = \chi(G^{(2)})$.

Injective colorings were first introduced by Hahn, Kratochvíl, Širáň, and Sotteau [14], where the authors showed injective colorings can be used in coding theory, by relating the injective chromatic number of the hypercube to the theory of error-correcting codes. The authors showed that for a graph G with maximum degree Δ, $\chi_i(G) \leq \Delta(\Delta - 1) + 1$. They also showed that computing the injective chromatic number is NP-complete and gave bounds and structural results for the injective chromatic numbers of graphs with special properties. It is easy to see that $\Delta(G) \leq \chi_i(G) \leq |V(G)|$.

For each $v \in V(G)$, let $L(v)$ be a set of colors assigned to v. Then $L = \{L(v)|v \in V(G)\}$ is a *list assignment* of G. Given a list assignment L of G, an injective coloring φ of G is called an *injective L-coloring* of G if $\varphi(v) \in L(v)$ for every $v \in V(G)$. A graph is injectively *k-choosable* if for any list assignment L, where $|L(v)| \geq k$ for all $v \in V(G)$, G has an injective L-coloring. The *injective choosability number* of G, denoted as $\chi_i^{\ell}(G)$, is the minimum k needed such that G is injectively k-choosable. It is clear that $\chi_i(G) \leq \chi_i^{\ell}(G)$.

Graphs with low injective chromatic numbers have been studied extensively. A number of authors have studied the injective chromatic number of graphs G in relation to their maximum degree, $\Delta(G)$, or their maximum average degree, $\text{mad}(G) = \max_{H \subseteq G}\{2|E(H)|/|V(H)|\}$, for instance [3–5,16]. As $\text{mad}(G) < \frac{2g(G)}{g(G)-2}$ for all planar graphs, we can compute a

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http://dx.doi.org/10.1016/j.disc.2017.05.014
0012-365X/

Table 1
Known results on the injective chromatic number and injective list chromatic number. A 'Yes' in the 'Planar' column indicates that the result holds only for planar graphs, and a 'No' indicates that the result holds for both planar and non-planar graphs. A * in the '$g(G)$' column indicates that the girth was obtained using the bound $\text{mad}(G) < \frac{2g(G)}{g(G)-2}$. Results in [9,10] are stated for injective coloring. However, the same proofs work also for injective list coloring (See Refs. [2,3,6,7,9,10,12,13,15,16]).

Bounds	Planar	$\Delta(G)$	mad(G)	$g(G)$	Authors
$\chi_i(G) \leq \Delta + 1$	Yes	≥ 18		≥ 6	Borodin and Ivanova [3]
$\chi_i(G) \leq \Delta + 3$	Yes			≥ 6	Dong and Lin [12]
$\chi_i(G) \leq \Delta + 3$	No		$< \frac{14}{5}$	$\geq 7^*$	Doyon, Hahn, and Raspaud [13]
$\chi_i(G) \leq \Delta + 4$	No		< 3	$\geq 6^*$	Doyon, Hahn, and Raspaud [13]
$\chi_i(G) \leq \Delta + 8$	No		$< \frac{10}{3}$	$\geq 5^*$	Doyon, Hahn, and Raspaud [13]
$\chi_i(G) \leq 5$	Yes	≤ 3		≥ 7	Lužar, Škrekovski, and Tancer [16]
$\chi_i^\ell(G) \leq \Delta + 1$	No		$< \frac{5}{2}$	$\geq 10^*$	Cranston, Kim and Yu [9]
$\chi_i^\ell(G) \leq \Delta + 1$	Yes	≥ 4		≥ 9	Cranston, Kim and Yu [9]
$\chi_i^\ell(G) = \Delta$	Yes	≥ 4		≥ 13	Cranston, Kim and Yu [9]
$\chi_i^\ell(G) = \Delta$	No		$< \frac{42}{19}$	$\geq 21^*$	Cranston, Kim and Yu [9]
$\chi_i^\ell(G) \leq 5$	No	≥ 3	$< \frac{36}{13}$	$\geq 8^*$	Cranston, Kim and Yu [10]
$\chi_i^\ell(G) \leq \Delta + 2$	No	≥ 4	$< \frac{14}{5}$	$\geq 7^*$	Cranston, Kim and Yu [10]
$\chi_i^\ell(G) \leq \Delta + 1$	Yes	≥ 24		≥ 6	Borodin and Ivanova [2]
$\chi_i^\ell(G) \leq \Delta + 2$	Yes	≥ 12		≥ 6	Li and Xu [15]
$\chi_i^\ell(G) \leq \Delta + 2$	Yes	≥ 8		≥ 6	Bu and Lu [6]
$\chi_i^\ell(G) \leq \Delta + 3$	Yes			≥ 6	Chen and Wu [7]
$\chi_i^\ell(G) \leq \Delta + 4$	Yes	≥ 30		≥ 5	Li and Xu [15]
$\chi_i^\ell(G) \leq \Delta + 5$	Yes	≥ 18		≥ 5	Li and Xu [15]
$\chi_i^\ell(G) \leq \Delta + 6$	Yes	≥ 14		≥ 5	Li and Xu [15]
$\chi_i^\ell(G) \leq 5$	Yes	≤ 3		≥ 6	This paper

bound for the girth of G, $g(G)$, given mad(G). Table 1 consists of results for the injective chromatic number and the injective choosability number of graphs which depend on planarity, the maximum degree, the maximum average degree, and the girth.

For a planar graph G with girth at least 6 and any maximum degree Δ, the best known result about the injective chromatic number is $\chi_i(G) \leq \Delta + 3$ [12]. In this paper, we improve this result for the case $\Delta = 3$. Moreover, we improve the result of Lužar, Škrekovski, and Tancer [16] by decreasing the girth condition and by changing to injective list coloring. We also improve the other two highlighted bounds in Table 1 in special cases.

Theorem 1. *Every planar graph G with $\Delta(G) \leq 3$ and $g(G) \geq 6$ is injectively 5-choosable.*

This theorem is a step towards the conjecture of Chen, Hahn, Raspaud, and Wang [8] that all planar subcubic graphs are injectively 5-colorable. In order to prove Theorem 1 we prove a slightly stronger result in Theorem 2. Let G be a graph and let L be a list assignment. A *precolored path* in G is a path P_k on k vertices where $|L(v)| = 1$ for all $v \in V(P_k)$ and there is at most one vertex $v \in V(P_k)$ with a neighbor in $G - P_k$. Moreover, $\deg_{P_k}(v)$ is maximal among the other vertices in P_k and v has at most one neighbor in $G - P_k$. Vertices with lists of size one are called *precolored*. The set of all precolored vertices $\mathcal{P}$ in G is *proper* if the lists of precolored vertices give a proper coloring of $G^{(2)}$ when restricted to $\mathcal{P}$. That is, the precolored vertices do not conflict among themselves.

Theorem 2. *Let G be a plane graph with $\Delta(G) \leq 3$ and $g(G) \geq 6$. Let $\mathcal{P} \subseteq V(G)$. Let L be a list assignment of G such that $|L(v)| \geq 5$ for $v \in V(G) \setminus \mathcal{P}$ and $|L(v)| = 1$ for $v \in \mathcal{P}$. If the precolored vertices are proper, are all in the same face, form at most two precolored paths, each of which is on at most three vertices, then G is injectively L-colorable.*

2. Preliminaries

The following notation shall be used in the sequel. A *k-vertex* is a vertex of degree k. We denote the degree of a vertex v by $\deg(v)$. We denote the set of neighbors of v by $N(v)$ and $N(v) \cup \{v\}$ by $N[v]$. If we want to stress that the degree or neighborhood is in a particular graph G, we use subscript G, e.g. $\deg_G(v)$. A *cut edge* or *bridge* is an edge which, when