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Limits of structures and the example of tree semi-lattices

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ABSTRACT

The notion of left convergent sequences of graphs introduced by Lovász et al. (in relation with homomorphism densities for fixed patterns and Szemerédi's regularity lemma) got increasingly studied over the past 10 years. Recently, Nešetřil and Ossona de Mendez introduced a general framework for convergence of sequences of structures. In particular, the authors introduced the notion of QF-convergence, which is a natural generalization of left-convergence. In this paper, we initiate study of QF-convergence for structures with functional symbols by focusing on the particular case of tree semi-lattices. We fully characterize the limit objects and give an application to the study of left convergence of m-partite cographs, a generalization of cographs.

1. Introduction

The study of limits of graphs gained recently a major interest [6,7,10,22–24]. In the framework studied in the aforementioned papers, a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is said *left-convergent* if, for every (finite) graph F, the probability

$$t(F, G_n) = \frac{\hom(F, G_n)}{|G_n|^{|F|}}$$

that a random map $f : V(F) \to V(G_n)$ is a *homomorphism* (i.e. a mapping preserving adjacency) converges as n goes to infinity. (For a graph G, we denote by $|G|$ the *order* of G, that is the number of vertices of G.) In this case, the limit object can be represented by means of a *graphon*, that is a measurable function $W : [0, 1]^2 \to [0, 1]$. The definition of the function t above is extended to graphons by

$$t(F, W) = \int \cdots \int \prod_{ij\in E(F)} W(x_i, x_j)\, \mathrm{d}x_1 \ldots \mathrm{d}x_p$$

(where we assume that F is a graph with vertex set $\{1, \ldots, p\}$) and then the graphon W is the left-limit of a left-convergent sequence of graphs $(G_n)_{n\in\mathbb{N}}$ if for every graph F it holds

$$t(F, W) = \lim_{n\to\infty} t(F, G_n).$$

For k-regular hypergraphs, the notion of left-convergence extends in the natural way, and left-limits – called *hypergraphons* – are measurable functions $W : [0, 1]^{2^k-2} \to [0, 1]$ and have been constructed by Elek and Szegedy using ultraproducts [11] (see also [29]). These limits were also studied by Hoover [15], Aldous [1], and Kallenberg [21] in the

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setting of exchangeable random arrays (see also [3]). For other structures, let us mention limits of permutations [17,16] and limits of posets [8,19,14].

A *signature* σ is a set of symbols of relations and functions with their arities. A σ-structure $\mathbf{A}$ is defined by its *domain* A and an interpretation in A of all the relations and functions declared in σ. A σ-structure is *relational* if the signature σ only contains symbols of relations. Thus relational structures are natural generalizations of k-uniform hypergraphs. To the opposite, a σ-structure is *functional* (or called an *algebra*) if the signature σ only contains symbols of functions. Denote by $\mathrm{QF}_p(\sigma)$ the fragment of all quantifier free formulas with p (free) variables (in the language of σ) and by $\mathrm{QF}(\sigma) = \bigcup_p \mathrm{QF}_p(\sigma)$ the fragment of all quantifier free formulas. In the following, we shall use QF_p and QF when the signature σ is clear from context. For a formula ϕ with p free variables, the set of satisfying assignments of ϕ is denoted by $\phi(\mathbf{A})$:

$$\phi(\mathbf{A}) = \{(v_1, \ldots, v_p) \in A^p : \mathbf{A} \models \phi(v_1, \ldots, v_p)\}.$$

In the general framework of finite σ-structures (that is a σ-structure with finite domain), the notion of *QF*-convergence has been introduced by Nešetřil and the third author [26]. In this setting, a sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of σ-structures is *QF-convergent* if, for every quantifier free formula ϕ with variables $x_1, \ldots, x_p$, the probability

$$\langle \phi, \mathbf{A}_n \rangle = \frac{|\phi(\mathbf{A}_n)|}{|A_n|^p} \tag{1}$$

that a random (uniform independent) assignment to the variables of ϕ of elements of A_n satisfies ϕ converges as n goes to infinity (this quantity $\langle \phi, \mathbf{A}_n \rangle$ will be referred to as the *Stone pairing* of ϕ and $\mathbf{A}_n$). Although originally defined for unweighted structures, these notions naturally extend to *weighted structures*, that is structures equipped with a non uniform probability measure.

The notion of QF-convergence extends several notions of convergence.

It was proven in [26] that a sequence of graphs (or of k-uniform hypergraphs) with order going to infinity is QF-convergent if and only if it is left-convergent. This is intuitive, as for every finite graph F with vertex set $\{1, \ldots, p\}$ there is a quantifier-free formula ϕ_F with variables $x_1, \ldots, x_p$ such that for every graph G and every p-tuple $(v_1, \ldots, v_p)$ of vertices of G it holds $G \models \phi_F(v_1, \ldots, v_p)$ if and only if the map $i \mapsto v_i$ is a homomorphism from F to G.

As mentioned before the left-limit of a left-convergent sequence of graphs can be represented by a graphon. However it cannot, in general, be represented by a Borel graph — that is a graph having a standard Borel space V as its vertex set and a Borel subset of $V \times V$ as its edge set. A graphon W is *random-free* if it is almost everywhere $\{0, 1\}$-valued. Notice that a random-free graphon is essentially the same (up to isomorphism mod 0) as a Borel graph equipped with a non-atomic probability measure on V. A class of graphs $\mathcal{C}$ is said to be *random-free* if, for every left-convergent sequence of graphs $(G_n)_{n\in\mathbb{N}}$ with $G_n \in \mathcal{C}$ (for all n) the sequence $(G_n)_{n\in\mathbb{N}}$ has a random-free limit.

Local convergence of graphs with bounded degree has been defined by Benjamini and Schramm [4]. A sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with maximum degree D is *local-convergent* if, for every $r \in \mathbb{N}$, the distribution of the isomorphism types of the distance r-neighborhood of a random vertex of G_n converges as n goes to infinity. This notion can also be expressed by means of QF-convergence (in a slightly stronger form). Let $G_1, \ldots, G_n, \ldots$ be graphs with maximum degree strictly smaller than D. By considering a proper edge coloring of G_n by D colors, we can represent G_n as a functional structure $\mathbf{V}_n$ with signature containing D unary functions $f_1, \ldots, f_D$, where V_n is the vertex set of G_n and $f_1, \ldots, f_D$ are defined as follows: for every vertex $v \in V_n$, $f_i(v)$ is either the unique vertex adjacent to v by an edge of color i, or v if no edge of color i is incident to v. It is easily checked that if the sequence $(\mathbf{V}_n)_{n\in\mathbb{N}}$ is QF-convergent if and only if the sequence $(G_n)_{n\in\mathbb{N}}$ of edge-colored graphs is local-convergent. If $(\mathbf{V}_n)_{n\in\mathbb{N}}$ is QF-convergent, then the limit is a *graphing*, that is a functional structure $\mathbf{V}$ (with same signature as $\mathbf{V}_n$) such that V is a standard Borel space, and $f_1, \ldots, f_D$ are measure-preserving involutions.

In the case above, the property of the functions to be involutions is essential. The case of quantifier free limits of general functional structures is open, even in the case of unary functions. Only the simplest case of a single unary function has been recently settled [27]. The case of QF-limits of functional structures with a single binary function is obviously at least as complicated as the case of graphs, as a graph G can be encoded by means of a (non-symmetric) function f defined by $f(u, v) = u$ if u and v are adjacent, and $f(u, v) = v$ otherwise, with the property that QF-convergence of the encoding is equivalent to left-convergence of the graphs. (Similarly, the case of a single k-ary function is at least as complex as the case of k-uniform hypergraphs.) The natural guess here for a limit object is the following:

Conjecture 1. *Let σ be the signature formed by a single binary functional symbol f.*

Then the limit of a QF-convergent sequence of finite σ-structures can be represented by means of a measurable function $w : [0, 1] \times [0, 1] \to \mathfrak{P}([0, 1])$, where $\mathfrak{P}([0, 1])$ stands for the space of probability measures on $[0, 1]$.

As witnessed by the case of local-convergence of graphs with bounded degrees, the "random-free" case, that is the case where the limit object can be represented by a Borel structure with same signature, is of particular interest. In this paper, we will focus on the case of simple structures defined by a single binary function – the tree semi-lattices – and we will prove that they admit Borel tree semi-lattices for QF-limits. Conversely, we will prove that every Borel tree semi-lattice (with domain equipped with an atomless probability measure) can be arbitrarily well approximated by a finite tree semi-lattice, hence leading to a full characterization of QF-limits of finite tree semi-lattices.