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A generating theorem of simple even triangulations with a finitizable set of reductions

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ABSTRACT

We shall determine exactly two (P, Q)-irreducible even triangulations of the projective plane. This result is a new generating theorem of even triangulations of the projective plane, that is, every even triangulation of the projective plane can be obtained from one of those two (P, Q)-irreducible even triangulations by a sequence of two expansions called a P-expansion and a Q-expansion, which were used in Batagelj (1984, 1989), Drapal and Lisonek (2010). Furthermore, we prove that for any closed surface F^2 there are finitely many (P, Q)-irreducible even triangulations of F^2.

1. Introduction

A *triangulation* G of a closed surface F^2 is a simple graph embedded on the surface such that each face of G is bounded by a 3-cycle. (A k-cycle means a cycle of length k.) A triangulation G is said to be *even* if each vertex of G has even degree. We do not regard K_3 on the sphere as a triangulation, and hence we can say that the smallest even triangulation of the sphere is the *octahedron*, which is the complete tripartite graph $K_{2,2,2}$ as an abstract graph. The neighbors of a vertex v of a triangulation form a cycle surrounding v on F^2, and we call it a *link* of v.

Especially in topological graph theory, we sometimes discuss *generating theorems* of graphs embedded on closed surfaces (i.e., constructing all graphs in a certain class $\mathcal{C}$ from $\mathcal{C}_0 \subset \mathcal{C}$ by a repeated application of certain expanding operations only through $\mathcal{C}$); e.g., $\mathcal{C}$ is a set of simple triangulations on the sphere and its well-known generating theorem will be mentioned later. Clearly, this notion is equivalent to that every graph in $\mathcal{C}$ can be reduced to one in $\mathcal{C}_0$ by a repeated application of the reductional operations, which are inverses of the above expanding operations; the set of such reductional operations is denoted by X here. In a generating theorem of graphs, $|X|$ and $|\mathcal{C}_0|$ are expected to be small; if possible, both of them should be finite. In particular, X is called *finitizable* for $\mathcal{C}$ if $|\mathcal{C}_0|$ is finite.

Let G be a simple triangulation on a closed surface. A *contraction* of e in G is to remove e, identify the two ends of e and replace two pairs of multiple edges by two single edges respectively. We say that e is *contractible* if the graph obtained from G by contracting e is simple. We say that a triangulation G is *irreducible* if G has no contractible edge. It follows from [2,6,12] that every surface admits only finitely many irreducible triangulations. That is, {contraction} is a finitizable set for simple triangulations on any closed surface. For fixed closed surfaces with low genera, the complete lists of irreducible triangulations had been obtained. Steinitz and Rademacher [13] had shown that a tetrahedron is the unique irreducible triangulation of the sphere. Those of the projective plane [1], the torus [10] and the Klein bottle [11,15] had been determined. (Furthermore, see Sulanke's results [14] for triangulations on closed surfaces with higher genera, using a computer.)

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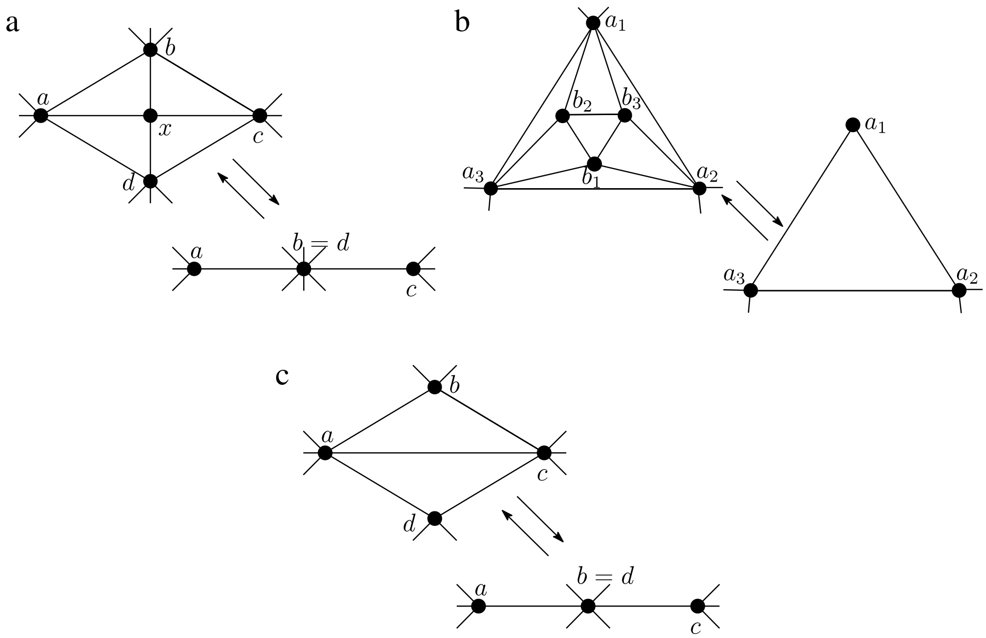

Fig. 1. Three local deformations: (a) P-reduction and P-expansion, (b) R-reduction and R-expansion, (c) Q-reduction and Q-expansion.

In this paper, we discuss generating theorems of simple even triangulations of closed surfaces by using some reductional (and expanding) operations defined as follows. (Throughout the paper, all even triangulations are assumed to be simple, unless otherwise specified. On the other hand, in [9,17], generating theorems of even triangulations with multiple edges are established, in order to solve some coloring problems inductively.)

Let G be an even triangulation on a closed surface F^2 and x be its vertex of degree 4 with a link $abcd$. A *P-reduction* of x at b, d is to eliminate a vertex x and identify two vertices b and d and replace two pairs of double edges by two single edges, respectively, as shown by (a) of Fig. 1. (In [4,5], one of b and d, say b here, is restricted to have degree 4. However, we do not add such a degree condition to b in this paper as well as [3,16].) If this operation yields multiple edges or loops, then we do not apply it. A vertex x of degree 4 in G is said to be *P-reducible* if we can apply a P-reduction of x. We call the inverse operation of a P-reduction a *P-expansion*.

Let G be an even triangulation and let f be a face of G bounded by $a_1a_2a_3$. An *R-expansion* to f is to put a 3-cycle $b_1b_2b_3$ into f and add edges a_ib_j for each distinct $i, j \in \{1, 2, 3\}$. In the resulting graph, the partial structure is isomorphic to the octahedron. Furthermore, we call the inverse operation of an R-expansion an *R-reduction*. These two operations are presented by (b) of Fig. 1. A facial 3-cycle C in G like $b_1b_2b_3$ is called *R-reducible* if removing C results in another even triangulation on the same surface.

Let G be an even triangulation and let abc and acd be two faces of G sharing an edge ac. A *Q-reduction* of ac (at $\{b, d\}$), is to eliminate an edge ac and identify two vertices b and d and replace two pairs of multiple edges by two single edges, respectively, as shown by (c) in Fig. 1. (Similar to P-reduction, b (or d) is restricted to have degree 4 in [4,5]. However, we do not add such a restriction to vertices since we consider even triangulations on general surfaces with higher genera; note that such an even triangulation might not have any vertex of degree 4.) If this operation yields multiple edges or loops, then we do not apply it. An edge ac in G is said to be *Q-reducible* if we can apply a Q-reduction of ac. An even triangulation with no Q-reducible edge is said to be *Q-irreducible*. We call the inverse operation of a Q-reduction a *Q-expansion*.

An even triangulation is said to be *(P, R)-irreducible* (resp., *(P, Q)-irreducible*) if it has no P-reducible vertex of degree 4 and no R-reducible 3-cycle (resp., no Q-reducible edge). Classically, Batagelj proved that the octahedron is the unique (P, R)-irreducible even triangulation of the sphere. This actually implies a generating theorem of even triangulations of the sphere as follows.

Theorem 1 (*Batagelj [3]*)**.** *The octahedron is the unique (P, R)-irreducible even triangulation of the sphere. Every even triangulation of the sphere can be obtained from the octahedron by a sequence of P-expansions and R-expansions.*

Similarly, Suzuki and Watanabe had discussed (P, R)-irreducible even triangulations on the projective plane and proved the following generating theorem.

Theorem 2 (*Suzuki, Watanabe [16]*)**.** *There are precisely 20 families of (P, R)-irreducible even triangulations of the projective plane. Every even triangulation of the projective plane can be obtained from one of those by a sequence of P-expansions and R-expansions.*