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A note on the linear 2-arboricity of planar graphs

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ABSTRACT

The linear 2-arboricity $\mathrm{la}_2(G)$ of a graph G is the least integer k such that G can be partitioned into k edge-disjoint forests, whose components are paths of length at most 2. In this paper, we prove that every planar graph G with $\Delta = 10$ has $\mathrm{la}_2(G) \le 9$. Using this result, we correct an error in the proof of a result in Wang (2016), which says that every planar graph G satisfies $\mathrm{la}_2(G) \le \lceil(\Delta+1)/2\rceil + 6$.

1. Introduction

Only simple and finite graphs are considered in this paper. For a graph G, we use $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$ (for short, Δ) to denote, respectively, its vertex set, edge set, minimum degree, and maximum degree. An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \ne j$. A *linear k-forest* is a graph whose components are paths of length at most k. The *linear k-arboricity* of G, denoted by $\mathrm{la}_k(G)$, is the least integer m such that G can be edge-partitioned into m linear k-forests.

The linear k-arboricity of a graph was first introduced by Habib and Péroche [5]. They posed the following conjecture, where $n = |V(G)|$ and $k \ge 2$:

$$\mathrm{la}_k(G) \le \begin{cases} \left\lceil \dfrac{n\Delta+1}{2\lfloor \frac{kn}{k+1} \rfloor} \right\rceil & \text{if } \Delta \ne n-1; \\ \left\lceil \dfrac{n\Delta}{2\lfloor \frac{kn}{k+1} \rfloor} \right\rceil & \text{if } \Delta = n-1. \end{cases}$$

The linear k-arboricity of graphs has been extensively investigated [1,3,4,6]. Lih, Tong and Wang [7] proved that every planar graph G has $\mathrm{la}_2(G) \le \lceil(\Delta+1)/2\rceil + 12$, and $\mathrm{la}_2(G) \le \lceil(\Delta+1)/2\rceil + 6$ if G moreover contains no 3-cycles. Recently, Wang [8] improved this result by showing that every planar graph G has $\mathrm{la}_2(G) \le \lceil(\Delta+1)/2\rceil + 6$. Unfortunately, there is a correctable error in the proof of the main result in [8], which has been found by the author herself, and independently by Dr. Xin Liu (a private communication). Precisely, the following statement in page 41, line 22, in [8] is not necessarily true:

"Similarly, we can show that $d_{F_2}(v_i) \le \beta(v_i)$."

In fact, when $d_G(v) = 15$, $d_{F_1'}(v_i) = 0$, and $d_{F_2'}(v_i) = 2$, we have $\beta(v_i) = \max\{2, \lceil(d_G(v_i) - 11)/2\rceil\} = 2$ and $d_{F_2}(v_i) = d_{F_2'}(v_i) + 1 = 3$, which does not satisfy that $d_{F_2}(v_i) \le \beta(v_i)$.

In this paper, we first show that every planar graph G with $\Delta = 10$ has $\mathrm{la}_2(G) \le 9$. Then, by using this result and adjusting the proof of Theorem 2 in [8], we will correct this error and keep original result unchanged.

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2. Planar graphs with $\Delta = 10$

Given a plane graph G, let $F(G)$ denote its face set. For $f \in F(G)$, we use $b(f)$ to denote the closed boundary walk of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk in clockwise order, with repeated occurrences of vertices allowed. A vertex of degree k (at most k, at least k) is called a *k-vertex* (k^--vertex, k^+-vertex). Similarly, we can define k-face, k^--face, and k^+-face. For a vertex $v \in V(G)$ and an integer $n \geq 1$, let $n_i(v)$ denote the number of i-vertices adjacent to v in G.

To investigate the linear 2-arboricity of a graph G, we consider the linear edge-coloring of G, i.e., a mapping ϕ from $E(G)$ to the color set C such that every color class induces a subgraph whose components are paths of length at most 2. We call such coloring *linear-k-coloring* of G if C contains k colors. Clearly, a graph G has linear 2-arboricity at most k if and only if G is linear-k-colorable.

A function L is called an assignment for the graph G if it assigns a list $L(e)$ of possible colors to each edge e of G. If G has a linear edge-coloring ϕ such that $\phi(e) \in L(e)$ for all edges e, then we say that ϕ is an L-linear edge-coloring of G, or say that G is L-linear edge-colorable.

Let S_m denote a star consisting of m edges $e_1, e_2, \ldots, e_m$, where $m \geq 2$. The following Lemma 1 was proved in [8].

Lemma 1 ([8]). *Let S_m be a star and L be a list assignment for the edges in S_m satisfying the following conditions, then S_m is L-linear edge-colorable.*

(1) $m = 2$, *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$.
(2) $m = 3$, *and* $|L(e_3)| \geq 2$ *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$.
(3) $m = 4$, *and* $|L(e_i)| \geq 2$ *for* $i = 3, 4$, *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$.
(4) $m = 5$, *and* $|L(e_5)| \geq 3$, $|L(e_i)| \geq 2$ *for* $i = 3, 4$, *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$.
(5) $m = 6$, *and* $|L(e_i)| \geq 3$ *for* $i = 5, 6$, $|L(e_i)| \geq 2$ *for* $i = 3, 4$, *and* $|L(e_i)| \geq 1$ *for* $i = 1, 2$.

Lemma 2 ([1]). *For any graph G,* $\mathrm{la}_2(G) \leq \Delta(G)$.

Theorem 3. *If G is a planar graph with $\Delta(G) \leq 10$, then* $\mathrm{la}_2(G) \leq 9$.

Proof. It suffices to prove that G has a linear-9-coloring. If $\Delta(G) \leq 9$, then the result holds automatically by Lemma 2. So assume that $\Delta(G) = 10$. The proof is proceeded by contradiction. Let G be a counterexample to the theorem such that $|V(G)| + |E(G)|$ is the least possible. So G is connected and $\delta(G) \geq 1$. For any proper subgraph H of G, H has a linear-9-coloring ϕ.

In the following, let $C = \{1, 2, \ldots, 9\}$ denote a set of nine colors. For a vertex $v \in V(H)$, we use $C(v)$ to denote the set of colors used in edges incident to v in H. Moreover, let $\Lambda(v)$ denote a sequence of colors in $C(v)$ some of which may appear twice. For example, $\Lambda(v) = (1, 1, 2, 3, 4, 5)$ indicates that v is a 6-vertex whose incident edges are colored with the colors $1, 2, \ldots, 5$, where 1 appears twice and the other appears only once. For an edge $xy \in E(G) \setminus E(H)$, let $L(xy) = C \setminus (C(x) \cup C(y))$, whose colors can apply to xy.

Let H be a largest component of the graph which is obtained from G by removing all 1-vertices and 2-vertices. Embed H into the plane. Then H is a connected plane graph with $\Delta(H) \leq 10$.

Claims 1–4 below can be proved similarly to the proof of Claims 1–4 in Theorem 7 in [8]. In fact, it is enough to replace 11 by 10 in some places.

Claim 1. *There is no edge $xy \in E(G)$ such that $d_G(x) + d_G(y) \leq 10$.*

Claim 2. *Let $v \in V(G)$ be a k-vertex with $5 \leq k \leq 10$ and $v_1, v_2, \ldots, v_k$ be the neighbors of v with $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$. Then the following statements hold.*

(0) $n_{11-k}(v) \leq 1$.
(1) *If* $n_{11-k}(v) = 1$, *then* $n_{12-k}(v) \leq 1$; *moreover, if* $n_{12-k}(v) = 1$, *then* $n_{13-k}(v) \leq 1$.
(2) $n_{11-k}(v) + n_{12-k}(v) \leq 3$; *moreover, if* $n_{11-k}(v) + n_{12-k}(v) = 3$, *then* $n_{13-k}(v) = 0$.
(3) *If* $k = 10$, *then* $n_1(v) + n_2(v) + n_3(v) \leq 5$; *moreover, if* $n_1(v) + n_2(v) \geq 3$, *then* $n_3(v) = 0$.

The following useful observation follows easily from Claims 1 and 2:

Observation 1. *Let $v \in V(H)$. Then following statements hold.*

(1) *If* $d_H(v) = 7$, *then* $d_G(v) = 7$; *or* $d_G(v) = 10$ *with* $n_2(v) = 3$.
(2) *If* $d_H(v) = 8$, *then* $d_G(v) = 8$; *or* $d_G(v) = 9$ *with* $n_2(v) = 1$; *or* $d_G(v) = 10$ *with* $n_1(v) + n_2(v) = 2$.
(3) *If* $d_H(v) = 9$, *then* $d_G(v) = 9$; *or* $d_G(v) = 10$ *with* $n_1(v) + n_2(v) = 1$.

Claim 3. $\delta(H) \geq 3$.

Claim 4. *If* $d_H(v) \leq 6$, *then* $d_H(v) = d_G(v)$.

For a vertex $v \in V(H)$ and an integer $i \geq 3$, let $n'_i(v)$ denote the number of i-vertices adjacent to v in H, and $m'_3(v)$ denote the number of 3-faces incident to v in H. By Claim 4, $n'_i(v) = n_i(v)$ for $i = 3, 4, 5$.