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Pebbling in semi-2-trees

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ABSTRACT

Graph pebbling is a network model for transporting discrete resources that are consumed in transit. Deciding whether a given configuration on a particular graph can reach a specified target is NP-complete, even for diameter two graphs, and deciding whether the pebbling number has a prescribed upper bound is Π_2^{P}-complete. Recently we proved that the pebbling number of a split graph can be computed in polynomial time. This paper advances the program of finding other polynomial classes, moving away from the large tree width, small diameter case (such as split graphs) to small tree width, large diameter, continuing an investigation on the important subfamily of chordal graphs called k-trees. In particular, we provide a formula, that can be calculated in polynomial time, for the pebbling number of any semi-2-tree, falling shy of the result for the full class of 2-trees.

1. Introduction

The fundamental question in graph pebbling is whether a given supply (*configuration*) of discrete pebbles on the vertices of a connected graph can satisfy a particular set of demands on the vertices. The operation of pebble movement across an edge $\{u, v\}$ is called a *pebbling step*: while two pebbles cross the edge, only one arrives at the opposite end, as the other is consumed. We write (u, v) to denote a pebbling step from u to v. The most studied scenario involves the demand of one pebble on a single *root* vertex r. Satisfying this demand is often referred to as *reaching* or *solving* r, and configurations are consequently called either r*-solvable* or r*-unsolvable*.

The *size* $|C|$ of a configuration $C : V \to \mathbb{N} = \{0, 1, \dots\}$ is its total number of pebbles $\sum_{v\in V} C(v)$. The *pebbling number* $\pi(G) = \max_{r\in V} \pi(G, r)$, where $\pi(G, r)$ is defined to be the minimum number s so that every configuration of size at least s is r-solvable. Simple sharp lower bounds like $\pi(G) \geq n$ and $\pi(G) \geq 2^{\mathrm{diam}(G)}$ are easily derived. Graphs satisfying $\pi(G) = n$ are called *Class* 0 and are a topic of much interest. Recent chapters in [13] and [12] include variations on the theme such as k-pebbling, fractional pebbling, optimal pebbling, cover pebbling, and pebbling thresholds, as well as applications to combinatorial number theory, combinatorial group theory, and p-adic diophantine equations, and also contain important open problems in the field.

Computing the pebbling number is difficult in general. The problem of deciding if a given configuration on a graph can reach a particular vertex was shown in [14] and [16] to be NP-complete, even for diameter two graphs [10] or planar graphs [15]. Interestingly, the problem was shown in [15] to be in P for graphs that are both planar and diameter two, as well as for outerplanar graphs (which include 2-trees). The problem of deciding whether a graph G has pebbling number at most k was shown in [16] to be Π_2^{P}-complete.

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In contrast, the pebbling number is known for many graphs. For example, in [17] the pebbling number of a diameter 2 graph G was determined to be n or $n+1$. Moreover, [9] and [4] characterized those graphs having $\pi(G) = n+1$, and it was shown in [11] that one can recognize such graphs in quartic time, improving on the order n^3m algorithm of [3]. Beginning a program to study for which graphs their pebbling number can be computed in polynomial time, the authors of [1] produced a formula for the family of split graphs that involves several cases. For a given graph, finding to which case it belongs takes $O(n^{1.41})$ time. The authors also conjectured that the pebbling number of a chordal graph of bounded diameter can be computed in polynomial time.

In opposition to the small diameter, large tree width case of split graphs, we turn here to chordal graphs with large diameter and small tree width.[1] Building on [2], in this paper we study 2-paths, the sub-class of 2-trees whose graphs have exactly two simplicial vertices, as well as what we call semi-2-trees, the sub-class of 2-trees, each of whose blocks are 2-paths, and prove an exact formula that can be computed in linear time.

2. Preliminary definitions and results

In order to simplify notation, for a subgraph $H \subset G$ or subset $H \subset V(G)$ we write $C(H)$ to denote $\sum_{v \in V(H)} C(v)$. We use C_H for the restriction of C to H.

A *simplicial* vertex in a graph is a vertex whose neighbors form a complete graph. It is *k-simplicial* if it also has degree k. A *k-tree* is a graph G that is either a complete graph of size k or has a k-simplicial vertex v for which $G - v$ is a k-tree. A *k-path* is a k-tree with exactly two simplicial vertices. A *semi-2-tree* is a graph in which each of its blocks is a 2-path, with each of its cut-vertices being simplicial in all of its blocks. For the purpose of our work we derive a new characterization of 2-paths that facilitates the analysis of its pebbling number.

Let $P = x_0, x_1, \dots, x_{d-1}, x_d$ be a shortest rs-path between two vertices $r = x_0$ and $s = x_d$ of G, where $d = \mathrm{dist}(r, s) = \mathrm{diam}(G)$. For $1 \le i \le d-1$, an *$x_{i-1}x_{i+1}$-fan* (*centered on x_i*) is a subgraph F of G consisting of the subpath x_{i-1}, x_i, x_{i+1} of P and a path $Q = x_{i-1}, v_{i,1}, \dots, v_{i,k_i}, x_{i+1}$ with $k_i \ge 1$ such that x_i is adjacent to every vertex of Q. We call F' the set $\{v_{i,1}, \dots, v_{i,k_i}\}$.

Let F_i be an $x_{i-1}x_{i+1}$-fan and F_{i+1} be an x_ix_{i+2}-fan, centered on x_i and on x_{i+1}, respectively. We say that F_i and F_{i+1} are *opposite-sided* if $F'_i \cap F'_{i+1} = \emptyset$; and that they are *same-sided* when $F'_i \cap F'_{i+1} = \{v_{i,k_i}\}$ and $v_{i,k_i} = v_{i+1,1}$.

The graph G is an *overlapping fan graph* if the following three conditions are satisfied:

- for every $1 \le i \le d-1$, there is a subgraph F_i which is an $x_{i-1}x_{i+1}$-fan centered on x_i,
- for every $1 \le i \le d-2$, F_i and F_{i+1} are either opposite-sided or same-sided, and
- G is the union of the subgraphs F_i for $1 \le i \le d-1$.

If we agree in calling F_1 an *upper* fan, then all further fans of an overlapping fan graph can be classified into upper or *lower* (opposite-sided from upper) — see Fig. 1.

Notice that, in general, the description of a graph as an overlapping fan graph, may be done using different paths P (see the examples in the center and right of Fig. 1). The path P used to describe G as an overlapping fan graph is called the *spine* of G.

In an overlapping fan graph, $|F'_i \cap F'_{i+3}| = 0$; while $|F'_{i-1} \cap F'_{i+1}| \le 1$, with equality if and only if $k_i = 1$. Notice that we can always choose the spine P so that $|F'_{i-1} \cap F'_{i+1}| = 0$ by swapping the names of vertices x_i and $v_{i,1}$, changing the fans F_{i-1}, F_i, and F_{i+1} from being same-sided to F_i being opposite-sided from F_{i-1} and F_{i+1}. Such a choice of path P is called *pleasant* (see Fig. 1).

For an internal vertex x_i of the spine of an overlapping fan graph G, we let A_{x_i} be the set of vertices of F'_i that are in no other fan of G. If $A_{x_i} = \emptyset$ then $k_i = 1$ and $v_{i,1} \in F'_{i-1}$ or F'_{i+1}; or $k_i = 2$ and $v_{i,1} \in F'_{i-1}$ and $v_{i,2} \in F'_{i+1}$. In the former let e_{x_i} be the edge $x_{i-1}v_{i,1}$ or $v_{i,1}x_{i+1}$ respectively, and in the latter let $e_{x_i} = \{v_{i,1}, v_{i,2}\}$. The following fact will be used in Section 5.2.

Claim 1. *If A_{x_i} is empty (non empty) then $G - e_{x_i}$ ($G - A_{x_i}$) is the union of two overlapping fan graphs each one with x_i as simplicial vertex and no other vertex in common.*

A 2-path of diameter 1 is just a path on two vertices. In this case, its *spine* is the graph itself. For larger diameter we have the following lemma.

Lemma 2. *A graph G of $\mathrm{diam}(G) \ge 2$ is a 2-path if and only if it is an overlapping fan graph.*

Proof. An overlapping fan graph is certainly a 2-path.

Let G be a 2-path with simplicial vertices r and s and diameter at least 2. The 2-path on 4 vertices is a fan, and hence an overlapping fan graph, so we assume that G has at least 5 vertices. Let $G' = G - s$, with simplicial vertices r and s'. Since G' is a 2-path, by induction it is also an overlapping fan graph.

If $\mathrm{diam}(G) > \mathrm{diam}(G')$ then the inclusion of s creates a new fan centered on s'. Otherwise, the inclusion of s extends the last fan of G'. In both cases, then, G is an overlapping fan graph. □

[1] One can find the definition of tree-width in [5], but it is not necessary for this paper.