Discrete Mathematics 340 (2017) 1503–1507

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Efficient dominating sets in circulant graphs ☆

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ARTICLE INFO

Article history:
Received 10 April 2016
Received in revised form 14 February 2017
Accepted 15 February 2017

Keywords:
Circulant graphs
Cayley graphs
Efficient dominating sets

ABSTRACT

Let S be a subset of finite cyclic group $\mathbb{Z}_n$ not containing the identity element 0 with $S = -S$. Cayley graphs on $\mathbb{Z}_n$ with respect to S are called circulant graphs and denoted by $\mathrm{Cay}(\mathbb{Z}_n, S)$. In this paper, for connected non-complete circulant graphs $\mathrm{Cay}(\mathbb{Z}_n, S)$ of degree $|S| = p - 1$ with p prime, we give a necessary and sufficient condition for the existence of efficient dominating sets, and characterize all efficient dominating sets if exist. We also obtain similar results for $\mathrm{Cay}(\mathbb{Z}_n, S)$ of degree $|S| = pq-1$ and p^m-1, where p, q are primes, m is a positive integer, and $|S|+1$ is relatively prime to $\frac{n}{|S|+1}$. Moreover, we give a necessary and sufficient condition for the existence of efficient dominating sets in $\mathrm{Cay}(\mathbb{Z}_n, S)$ of order $n = p^kq, p^2q^2, pqr, p^2qr, pqrs$ and degree $|S|$, where p, q, r, s are distinct primes, k is a positive integer, and $|S| + 1$ is relatively prime to $\frac{n}{|S|+1}$.

1. Introduction

Let G be a finite group with identity element e and S a subset of G not containing e with $S = S^{-1}$. The *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined as the graph with vertex set G and edge set $\{\{g, sg\} : g \in G, s \in S\}$. Clearly, $\mathrm{Cay}(G, S)$ is a graph of degree $|S|$, and $\mathrm{Cay}(G, S)$ is connected if and only if G is generated by S, which is denoted by $G = \langle S\rangle$. Cayley graphs on finite cyclic group $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$ with addition modulo n are called *circulant graphs* and denoted by $\mathrm{Cay}(\mathbb{Z}_n, S)$. Moreover, we denote the identity of $\mathbb{Z}_n$ by 0 and denote $S \cup \{0\}$ by S_0.

A subset D of vertices in a graph Γ is called an *efficient dominating set* (also called a *perfect code*) if D is an independent set and each vertex not in D is adjacent to exactly one vertex in D. By the definitions of circulant graph and efficient dominating set, it is easy to see that $n = |S_0||D|$ if $\mathrm{Cay}(\mathbb{Z}_n, S)$ admits an efficient dominating set D.

The problem of determining the existence of efficient dominating sets in some families of graphs was first investigated by Biggs [1] and Kratochvíl [8]. Later Livingston and Stout [11] studied the existence and construction of efficient dominating sets in families of graphs arising from the interconnection networks of parallel computers. Lee [10] proved that a Cayley graph on an abelian group admits an efficient dominating set if and only if it is a covering graph of a complete graph. Knor and Potočnik [7] proved that connected cubic vertex-transitive graph on a power of 2 vertices admits an efficient dominating set if and only if it is not isomorphic to a Möbius ladder. Recently, the problem of determining the existence of efficient dominating sets in circulant graphs has been studied by several researchers. For example, Tamizh Chelvam and Mutharasu [13] gave a necessary and sufficient condition for a subgroup to be an efficient dominating set in circulant graphs. Kumar and MacGillivray [9] characterized the efficient dominating sets in $\mathrm{Cay}(\mathbb{Z}_n, S)$ with $\frac{n}{|S_0|} = 2$ and 3, and Deng [4] further generalized this result by considering the case of $\frac{n}{|S_0|} = p$ with p prime. For more results regarding efficient dominating set, we refer the reader to [2,3,6].

☆ This research is supported by National Natural Science Foundation of China (No. 11401368), Shanghai Municipal Natural Science Foundation (No. 14ZR1417900) and Introduction of Shanghai University of Electric Power Scientific Research Grants Project (No. K2013-006).

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http://dx.doi.org/10.1016/j.disc.2017.02.014
0012-365X/

In [12], Obradović, Peters and Ružić gave necessary and sufficient conditions for the existence of efficient dominating sets in connected circulant graphs of degree 3 and 4. The present paper generalizes the results in [12]. For connected non-complete circulant graphs $\mathrm{Cay}(\mathbb{Z}_n, S)$ of degree $|S| = p-1, pq-1$ and p^m-1 (p, q primes, m positive integer), we give necessary and sufficient conditions for the existence of efficient dominating sets completely in terms of the generating set S. Moreover, for connected circulant graphs of order $n = p^kq, p^2q^2, pqr, p^2qr, pqrs$ (p, q, r, s distinct primes, k positive integer), we also obtain a necessary and sufficient condition for the existence of efficient dominating sets. It is deserved to be mentioned that in [5], Feng etc. gave necessary and sufficient conditions for the existence of efficient dominating sets in connected circulant graphs of degree p^l-1 (p prime and l positive integer). The main tool in [5] is cyclotomic polynomials, and the main tool in this paper is the subset direct sum method introduced by us in [4] for dealing with the existence of efficient dominating sets in Cayley graphs on abelian groups. The rest of the paper is organized as follows. In Section 2, we gather some definitions and known results needed later. In Section 3, we present and prove our main results.

2. Preliminaries

Let G be a finite abelian group with additive notation. We define the sum of two subsets M, N of G by $M+N = \{m+n : m \in M, n \in N\}$. If M contains only one element m, we write $m+N$ instead of $\{m\}+N$. Two subsets M, N of G are called *supplementary* if each $g \in G$ has a unique representation in the form $g = m+n$ with $m \in M$ and $n \in N$. In this case, we write $G = M \oplus N$.

Proposition 2.1 (*[14], Proposition 2.1, p. 32*)**.** *Let G be a finite abelian group and let M, N be subsets of G. Then $G = M \oplus N$ is equivalent to the conjunction of any two of the following conditions*: (i) $G = M+N$; (ii) $(M-M)\cap(N-N) = \{0\}$; (iii) $|G| = |M||N|$.

For any subset $M \subseteq G$, the set of all $g \in G$ satisfying $g+M = M$ is a subgroup of G, called the *stability subgroup* of M, denoted by G_M. A subset $M \subseteq G$ is called *periodic* if $G_M \neq \{0\}$. Similarly, a subset $M \subseteq G$ is called *aperiodic* if $G_M = \{0\}$. It is easy to see that M is a union of cosets of G_M, and thus $|M|$ is divisible by $|G_M|$. Let $M/G_M = \{g+G_M : g \in M\}$. If $0 \in M$, then $G_M \subseteq M$ and the subset $M/G_M \subseteq G/G_M$ is aperiodic.

Before stating the following two propositions, following [14], we introduce some special sets of integers: $N_0 = \{p^k : p$ prime, $k \geq 0\}$; $N_1 = \{p^kq : p, q$ distinct primes, $k \geq 1\}$; $N_2 = \{p^2q^2 : p, q$ distinct primes $\}$; $N_3 = \{p^kqr : p, q, r$ distinct primes, $k \in \{1,2\}\}$; $N_4 = \{pqrs : p, q, r, s$ distinct primes$\}$; $\overline{N} = \bigcup_{i=0}^{4} N_i$.

Proposition 2.2 (*[14], Theorem 2.1, p. 33*)**.** *For every integer $m \geq 1$ the following conditions are equivalent*: (i) $m \in N_0$; (ii) *For every integer $n \geq 2$ and every pair M, N of supplementary subsets of $\mathbb{Z}_n$ with $|M| = m$, it is true that at least one of the subsets M, N is periodic.*

Proposition 2.3 (*[14], Theorem 2.2, p. 33*)**.** *For every integer $n \geq 2$ the following conditions are equivalent*: (i) $n \in \overline{N}$; (ii) *In every pair of supplementary subsets of $\mathbb{Z}_n$, at least one of the subsets is periodic.*

Lemma 2.4 (*[4]*)**.** *A subset D of finite abelian group G is an efficient dominating set of $\mathrm{Cay}(G, S)$ if and only if $G = S_0 \oplus D$.*

Lemma 2.5 (*[4]*)**.** *Let $0 \in S_0$ and D be two subsets of finite abelian group G and set $A := G_{S_0}$. Then $G = S_0 \oplus D$ if and only if $G/A = S_0/A \oplus (D+A)/A,\ A\cap(D-D) = \{0\}$.*

The following result first appears in [12] as Remark 1, and also appears in [4].

Lemma 2.6 (*[12], Remark 1, p. 258*)**.** *Let $n = |S_0|d$, where d is an integer. Then $\mathrm{Cay}(\mathbb{Z}_n, S)$ admits an efficient dominating set D which is a coset of some subgroup of $\mathbb{Z}_n$ if and only if $s-s' \not\equiv 0 \pmod{|S_0|}$ for any distinct $s, s' \in S_0$.*

Lemma 2.7 (*[4]*)**.** *Let $n = |S_0|p$, p prime, S_0 is aperiodic. Then $\mathrm{Cay}(\mathbb{Z}_n, S)$ admits an efficient dominating set if and only if $s-s' \not\equiv 0 \pmod{|S_0|}$ for any distinct $s, s' \in S_0$.*

3. Main result

In this section, we consider non-complete and non-empty graphs, since the efficient dominating sets in complete graphs and empty graphs have been well known. Moreover, in view of the fact that a non-connected graph admits an efficient dominating set if and only if its each component admits an efficient dominating set, we only consider connected graphs.

Theorem 3.1. *Let n be a positive integer, let p be a prime such that p divides n, let S be a subset of $\mathbb{Z}_n\backslash\{0\}$ of cardinality $p-1$ such that $S = -S$, and let $S_0 = S\cup\{0\}$. If a circulant graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ is connected and non-complete, then $\mathrm{Cay}(\mathbb{Z}_n, S)$ admits an efficient dominating set if and only if $s-s' \not\equiv 0 \pmod{p}$ for any distinct $s, s' \in S_0$. Furthermore, if $\mathrm{Cay}(\mathbb{Z}_n, S)$ admits an efficient dominating set, then the set $\mathbb{D}$ of all efficient dominating sets of $\mathrm{Cay}(\mathbb{Z}_n, S)$ is $\{g+p\mathbb{Z}_n : g \in \mathbb{Z}_n\}$.*