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The minimum feedback arc set problem and the acyclic disconnection for graphs[☆]

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ABSTRACT

A minimum feedback arc set of a digraph D is a minimum set of arcs which removal leaves the resultant graph free of directed cycles; its cardinality is denoted by $\tau_1(D)$. The acyclic disconnection of D, $\overrightarrow{\omega}(D)$, is defined as the maximum number of colors in a vertex coloring of D such that every directed cycle of D contains at least one monochromatic arc. In this article we study the relationship between the minimum feedback arc set and the acyclic disconnection of a digraph, we prove that the acyclic disconnection problem is $\mathcal{NP}$-complete. We define the acyclic disconnection and the minimum feedback for graphs. We also prove that $\overrightarrow{\omega}(G) + \tau_1(G) = |V(G)|$ if G is a wheel, a grid or an outerplanar graph.
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1. Introduction and primary results

A *minimum feedback arc set* of a digraph D is a minimum set of arcs which, if removed, leaves the resultant graph free of directed cycles; its cardinality is denoted by $MFAS(D)$ or $\tau_1(D)$. Finding a minimum feedback arc set is classical problem in combinatorial optimization [1,3–5,7,8]. Due to its many applications, efficient heuristics have been proposed to solve this problem, cf. [3,5]. However, it is not an easy task to calculate $\tau_1(D)$, the minimum feedback arc set problem is one of Karp's original 21 $\mathcal{NP}$-complete problems, and it remains $\mathcal{NP}$-hard even for tournaments and bipartite tournaments [1,4].

The *acyclic disconnection*, $\overrightarrow{\omega}(D)$, is defined as the maximum number of colors in a vertex coloring of D for which every directed cycle of D contains at least one monochromatic arc. The definition and first results about the behavior of this parameter in general digraphs were introduced by Neumann-Lara in [12]. It is worth noticing that this parameter is related with the dichromatic number, also introduced by Neumann-Lara in [11] and independently by Mohar in [10], in the following way. While for the dichromatic number we look for colorings such that every chromatic class is acyclic, for the acyclic disconnection we look for colorings such that the set of arcs not contained in a chromatic class induces an acyclic digraph.

The acyclic disconnection has been studied for special families of digraphs, such as tournaments [9,12] and multipartite tournaments [6]. There are not many articles relating the acyclic disconnection with other parameters, one example is [2], where its relation with the directed girth is explored. Although the acyclic disconnection was introduced in 1999, the computational complexity of the associated decision problem was unknown until now. In Section 3 we prove that the acyclic disconnection problem is $\mathcal{NP}$-complete. In Section 4 we discuss ways to relate $\overrightarrow{\omega}(D)$ with $\tau_1(D)$.

We can define these two parameters for graphs as follows: Let G be a graph and $\mathcal{O}(D)$ the set of its orientations. Define $\tau_1(G)$ and $\overrightarrow{\omega}(G)$ as $\tau_1(G) = \max_{D\in\mathcal{O}(D)}\tau_1(D)$ and $\overrightarrow{\omega}(D) = \min_{D\in\mathcal{O}(D)}\overrightarrow{\omega}(D)$. In general there is not an obvious relation between

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these two parameters, but, in Sections 5 and 6, we prove that $\tau_1(G) + \overrightarrow{\omega}(G) = |V(G)|$ for three families of planar graphs: wheels, grids and outerplanar graphs.

2. Notation and preliminaries

We use the notation of [3]. In particular, $(X_1, X_2)_D$ denotes the set of arcs with tail in X_1 and head in X_2. The number of weak components of a digraph D (the number of connected components of the underlying graph) is denoted by $\omega(D)$. If D_1 and D_2 are digraphs, $D_1 \square D_2$ denotes the usual cartesian product of digraphs. If u is a vertex of D_1 then D_2^u is the u-fiber of $D_1 \square D_2$; analogously for v in $V(D_2)$.

Let Γ_s denote the set of colors $\{1, 2, \ldots, s\}$. If D is a digraph and $\varphi : V(D) \to \Gamma_s$ is a vertex coloring of D, then the coloring φ induces two natural spanning subdigraphs of D. The *monochromatic digraph* of D, $M_\varphi(D)$, is the spanning subdigraph of D with arc set $\{uv \in A(D) : \varphi(u) = \varphi(v)\}$ and the *heterochromatic digraph* of D, $H_\varphi(D)$, the spanning subdigraph of D with arc set $\{uv \in A(D) : \varphi(u) \neq \varphi(v)\}$. We say that a coloring $\varphi : V(D) \to \Gamma_s$ is *externally acyclic* if $H_\varphi(D)$ is an acyclic digraph. The following observation follows from Proposition 5 of [6] and the definition of the acyclic disconnection of D.

Proposition 1. *Let D be a digraph. Then $\overrightarrow{\omega}(D) \geq s$ if and only if there is a proper coloring φ of D with s colors such that $H_\gamma(D)$ is acyclic, i.e., φ is an externally acyclic coloring of D.*

It is well known that D is acyclic if and only if there exists an ordering of its vertex set $V(D) = \{v_0, v_1, \ldots, v_n\}$ such that v_0 is a source of D and for every $1 \leq k < n$, v_k is a source of $D \setminus \{v_0, v_1, \ldots, v_{k-1}\}$. Such an order is called a *source ordering* of D.

3. $\mathcal{NP}$-completeness

In this section we prove that the acyclic disconnection problem is $\mathcal{NP}$-hard by reduction from the minimum feedback arc set problem. For this purpose we state the following two problems.

Minimum Feedback Arc Set (FAS)
Instance: A digraph D and a nonnegative integer k.
Question: Is $\tau_1(D) \leq k$?

Acyclic Disconnection (AD)
Instance: A digraph D and a nonnegative integer s.
Question: Is $\overrightarrow{\omega}(D) \geq s$?

Theorem 2. *The acyclic disconnection problem is $\mathcal{NP}$-complete, even for bipartite oriented graphs.*

Proof. Problem AD is in $\mathcal{NP}$ since a short certificate for a digraph D with a coloring φ with s colors is the digraph $H_\varphi(D)$. We can verify in polynomial time whether this digraph is acyclic, which implies by Proposition 1 that $\overrightarrow{\omega}(D) \geq s$.

To show completeness, we present a polynomial reduction from FAS. Given an instance (D, k) of FAS, we construct, in polynomial time, an instance (D', s), with $s = n - k$, of AD such that $\tau_1(D) \leq k$ if and only if $\overrightarrow{\omega}(D') \geq s$.

Let D be a digraph with $V(D) = \{v_1, v_2, \ldots, v_n\}$. Without loss of generality, we can assume that D is a tournament, [4]. Let D' be the digraph obtained by subdividing every arc of D. That is, $V(D') = V(D) \cup \{v_{xy} : xy \in A(D)\}$ and we replace every arc xy of D with the path $xv_{xy}y$.

Given an optimal externally acyclic coloring of D', $\varphi : V(D') \longrightarrow \Gamma_s$, we construct a feedback arc set S of D with $n - s$ arcs in order to show that $\tau_1(D) \leq n - s = k$.

If a monochromatic set X of an externally acyclic coloring induces a disconnected graph, then one of the connected components of $D'[X]$ can be colored with a new color without losing the property of being externally acyclic. Since φ is an optimal externally acyclic coloring, every subdigraph induced by a monochromatic set of $V(D')$ is connected.

Suppose, for a contradiction, that there exists a monochromatic path $xv_{xy}y$ in D'. Define $N = \{v \in N^-_{D'}[y] \mid \varphi(v) = \varphi(y)\}$. Then, we can define a new coloring $\sigma : V(D') \longrightarrow \Gamma_{s+1}$ such that

$$\sigma(u) = \begin{cases} s+1 & \text{if } u \in N \\ \varphi(u) & \text{otherwise.} \end{cases}$$

Observe that σ is onto because $\sigma(x) = \varphi(y)$ and $x \notin N$. The new coloring σ cannot be externally acyclic because φ is optimal. Therefore, there exists a directed cycle C of $H_\sigma(D')$ which is not a cycle of $H_\varphi(D')$. Then, there exists an arc a of C such that $a \in A(M_\varphi(D')) \cap A(H_\sigma(D'))$, thus $a = wv_{wy}$, for some $w \in N^-_D(y)$, because these are the only arcs with that property. So, $v_{wy}y \in A(M_\sigma(D')) \cap A(C)$ because y is the only out-neighbor of v_{wy}, which contradicts that C is a cycle of $H_\sigma(D')$. Hence, we conclude that the three vertices of a subdivision of an arc of D use at least two colors in D'.

It follows that every nontrivial subdigraph induced by a monochromatic class of φ is a star with a vertex of D as the center. Let S be the set of arcs of D whose subdivision has a monochromatic arc of D' with the coloring φ. Since the subdivision of every directed cycle of D is a directed cycle of D', every directed cycle of D has an arc in S and, therefore S is a feedback arc set of D. Observe that $|V(D')|$ and $|A(D')|$ are linear in terms of $|V(D)|$ and $|A(D)|$.

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