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# An approximate ore-type result for tight Hamilton cycles in uniform hypergraphs

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ABSTRACT

A Hamilton $\ell$-cycle in a $k$-uniform hypergraph of $n$-vertex is an ordering of all vertices, combined with an ordered subset $C$ of edges, such that any two consecutive edges share exactly $\ell$ vertices and each edge in $C$ contains $k$ consecutive vertices.

A classic result of O. Ore in 1960 is that if the degree sum of any two independent vertices in an $n$-vertex graph is at least $n$, then the graph contains a Hamiltonian cycle. Naturally, we consider to generalize it to hypergraphs situation. In this paper, we prove the following theorems. (i) For any $n \geq 4k^2$, there is an $n$-vertex $k$-uniform hypergraph, with degree sum of any two strongly independent sets of $k-1$ vertices bigger than $2n-4(k-1)$, contains no Hamilton $l$-cycle, $1 \leq \ell \leq k-1$. (ii) If the degree sum of two weakly independent sets of $k-1$ vertices in an $n$-vertex $k$-uniform hypergraph is $(1+o(1))n$, then the hypergraph contains a Hamilton $(k-1)$-cycle, where two distinct sets of $k-1$ vertices are weakly (strongly) independent if there exist no edge containing the union of them (intersecting both of them).



## 1. Introduction

A $k$-uniform hypergraph is pair $H = (V, E)$ of a vertex set $V$ and an edge set $E$, where each edge contains precisely $k$ vertices. A Hamilton $\ell$-cycle $C$ in $H$ is an ordered subset $\{e_1, e_2, \ldots, e_{|C|}\}$ of $E$ such that for some cyclic ordering of $V$, every edge in $C$ consists of $k$ consecutive vertices of the ordering and for every pair of consecutive edges $e_i, e_{i+1}$ in $C$, we have that $|e_i \cap e_{i+1}| = \ell$. Furthermore, let $\binom{V}{k-1}$ denote the family of all $k-1$ vertices set in $V$, and for any $S \in \binom{V}{k-1}$, denote $\deg(S) = |\{e \in E : S \subset e\}|$ the degree of $S$.

Two of the classic theorems in Hamiltonian graphs are proved by G.A. Dirac and O. Ore. In 1952, Dirac [1] proved that any $n$-vertex graphs with minimum degree at least $\frac{n}{2}$, $n \geq 3$, is Hamiltonian. Later in 1960, Ore [7] showed that a graph $G$ with $\sigma_2(G) \geq n$ is Hamiltonian, here $\sigma_2(G) = \min\{d(u) + d(v) : u, v \text{ are independent}\}$.

It is natural to generalize these two classic theorems to $k$-uniform hypergraphs. In 1999, a natural extension of Dirac's theorem has been conjectured by Katona and Kierstead [5] and approximately solved in the case $k \geq 3$ by V. Rödl, A. Ruciński and E. Szemerédi [8] as follows.

**Theorem 1.1** (Rödl, Ruciński and Szemerédi [8]). *Let $H$ be a $k$-uniform hypergraph, and let $\delta_{k-1}(H)$ be the minimum of* $\deg(S)$ *over all $S \in \binom{V}{k-1}$. For every $k \geq 3$ and $\gamma > 0$ there exists an $n_0$ such that every $k$-uniform $n$-vertex hypergraph with $\delta_{k-1}(H) \geq (\frac{1}{2} + \gamma)n$, $n \geq n_0$, contains a Hamilton $(k-1)$-cycle.*

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They later strengthen their result for 3-uniform hypergraph, which solved the conjecture for the case $k = 3$ [9].

In [8], they proposed a very powerful method, named absorbing, for Hamilton cycle in hypergraphs. Roughly speaking, they divided constructing a Hamilton cycle into four steps. First, one may construct a short path $P$ that can absorb any vertex set with small size. Second, find a reservoir set $R$. Then, prove there exists a collection $\mathcal{P}$ of vertex-disjoint paths covering almost all the remained vertex. At last, connect all the paths in $\mathcal{P}$ and $P$ by only using the vertex in $R$, and absorb the uncovered vertices into $P$.

Based on their approach, H. Hàn and M. Schacht [3] later proved an approximately best possible result that for any $1 \le \ell < \frac{k}{2}$, every $n$-vertex $k$-uniform hypergraph $H$ with $\delta_{k-1}(H) = (\frac{1}{2(k-\ell)} + o(1))n$ contains a Hamilton $\ell$-cycle. The best result for $1 \le \ell \le k$, proved by J. Han and Y. Zhao [2], is that any $n$-vertex $k$-uniform hypergraph $H$ such that $n$ is sufficiently large and $\delta_{k-1}(H) \ge \lceil \frac{n}{2k-2\ell} \rceil$ contains a Hamilton $\ell$-cycle. For more general case when $1 \le \ell < k$, Kühn, Mycroft and Osthus [6] proved that when $(k-\ell) \nmid k$, any $n$-vertex $k$-uniform hypergraph with $\delta_{k-1}(H) = (\frac{1}{\lceil \frac{k}{k-l} \rceil (k-\ell)} + o(1))n$ contains a Hamilton $\ell$-cycle.

In order to study the Ore-type condition for Hamiltonian cycle, we consider two types of independent sets between a pair of $(k-1)$-vertex sets stated in the following definition, which are generalizations of independence in graphs.

**Definition 1.1.** Given $S_1, S_2 \in \binom{V}{k}$, we say $S_1, S_2$ are strongly independent if there exists no such edge $e$, that $e \cap S_1 \neq \emptyset$ and $e \cap S_2 \neq \emptyset$; $S_1, S_2$ are weakly independent if there exists no such edge $e$ that $S_1 \cup S_2 \subset e$.

Note that two strongly independent sets are also weakly independent. For a $k$-uniform hypergraph $H$, define $\sigma_{2'}^{k-1}(H) = \min\{\deg(S) + \deg(T) : S, T \in \binom{V}{k-1}$ and $S, T$ are strongly independent$\}$, $\sigma_2^{k-1}(H) = \min\{\deg(S) + \deg(T) : S, T \in \binom{V}{k-1}$ and $S, T$ are weakly independent$\}$. We show that only bound $\sigma_{2'}^{k-1}(H)$ is not sufficient to ensure $H$ to be Hamiltonian.

**Theorem 1.2.** *For any $n \ge 4k^2$, $k \ge 3$, there exist an $n$-vertex $k$-uniform hypergraph $J_n$ with $\sigma_{2'}^{k-1}(J_n) \ge 2n - 4(k-1)$ that contains no Hamilton $\ell$-cycle, where $1 \le \ell \le k-1$.*

When $\sigma_2^{k-1}(H) \ge (1+o(1))n$, note that if there is a $(k-1)$-tuple $Q$ with degree less than $(\frac{1}{2}+o(1))n$, then every $(k-1)$-tuple in $V(H)\backslash Q$ has high degree, and thus by the result of [8], there is a tight cycle covering $V(H)\backslash Q$. In fact, it could be a tight cycle missing only two vertices from $Q$. We show that $\sigma_2^{k-1}(H) = (1+o(1))n$ is enough for $H$ to contain a Hamilton $(k-1)$-cycle.

**Theorem 1.3.** *For every $k \ge 3$, $\gamma > 0$, there exist $n_0$ such that every $n \ge n_0$ vertices $k$-uniform hypergraph $H$, $n \ge n_0$, with $\sigma_2^{k-1}(H) \ge (1+\gamma)n$, contains a Hamilton $(k-1)$-cycle.*

This bound is approximately best possible since it was shown in [5] that there exists a $k$-uniform hypergraph $H$ on $n$ vertices with $\delta_{k-1}(H) \ge \lceil \frac{n}{2} \rceil - 1$, which contains no Hamilton $(k-1)$-cycle.

## 2. Proof of Theorem 1.2

We construct the required hypergraph as follows. Let $J = (V, E)$ be a $k$-uniform hypergraph on $n$ vertices, where $V = A \cup B \cup C$, $|A| = |C| = k-1$, $|B| = n - 2(k-1)$, and every edge $e$ spans between $A$ and $B$ or between $C$ and $B$, i.e.,

(i) $e \cap B \neq \emptyset$, $e \cap A \neq \emptyset$ and $e \cap C = \emptyset$,
or (ii) $e \cap B \neq \emptyset$, $e \cap C \neq \emptyset$ and $e \cap A = \emptyset$.

**Proof of Theorem 1.2.** First, it is easy to check that $A, C$ are strongly independent in $J$. We next show that every pair $S_1, S_2 \in \binom{V}{k-1}$, $S_1 \neq S_2$, are not strongly independent in $J$ except for $S_1 = A$ and $S_2 = C$. Thus, $\sigma_{2'}^{k-1}(J) = \deg(A) + \deg(C) = 2(n - 2(k-1)) = 2n - 4(k-1)$.

Let $S_1 \neq A$ and $S_1 \neq C$.

**Case 1** $S_1 \cap B \neq \emptyset$.

Then let $e = \{u\} \cup \{v\} \cup \{R\}$, where $u \in S_1 \cap B$, $v \in S_2 \setminus S_1$ and $R \subset A \setminus \{u\}$, $|R| = k-2$. It is clear that $S_1$ and $S_2$ is not strongly independent under this $e$.

**Case 2** $S_1 \cap B = \emptyset$.

We have that $S_1 \cap A \neq \emptyset$ and $S_1 \cap C \neq \emptyset$. Note that by the discussion in Case 1, if $S_2 \cap B \neq \emptyset$, we also have that $S_1$ and $S_2$ are not strongly independent. However if $S_2 \cap B = \emptyset$, then $S = S_2 \cap A \neq \emptyset$ or $T = S_2 \cap C \neq \emptyset$. Hence, let $e_1 = A \cup \{u\}$ and $e_2 = C \cup \{u\}$, where $u \in B$, then we conclude $(e_1 \cap S_1) \cup (e_1 \cap S_2) \neq \emptyset$ or $(e_2 \cap S_1) \cup (e_2 \cap S_2) \neq \emptyset$. Thus $S_1$ and $S_2$ are not strongly independent.

We now prove that $J$ contains no Hamilton $l$-cycle, $1 \le \ell \le k-1$. Otherwise $V$ can be ordered cyclically and there is a collection of edges $C$ such that the edges $e_i$ in $C$ corresponds to some consecutive $k$ vertices and $|e_i \cap e_{i+1}| = \ell$.

We claim that every consecutive $2k-2$ vertices $L = \{v_1, v_2, \ldots, v_{2k-2}\}$ in this order must contain an edge in $C$. Suppose that $e_{i_0}$ is the last edge that meet $v_1$, i.e., $v_1 \notin e_{i_0+1}$ and $v_1 \in e_{i_0}$, and let $v_s$ be the last vertex in $e_{i_0}$. It is easy to see that $s \le k$, and by definition of Hamilton $\ell$-cycle, $|e_{i_0+1} \cap e_{i_0}| = \ell$. Thus $e_{i_0+1} = \{v_{s-\ell+1}, v_{s-\ell+2}, \ldots, v_{s-\ell+k}\} \subset L$.

Finally, we show that there exist some consecutive $2k-2$ vertices $L'$ that avoid $A$ and $C$, thus there is an edge in $L'$ consists of vertices in $B$, which contradicts the construction of $E$. We partition $\{v_1, v_2, \ldots, v_n\}$ into segments of length $2k-2$, i.e., we set $L_i = \{v_{(2k-2)i+1}, v_{(2k-2)i+2}, \ldots, v_{(2k-2)(i+1)}\}$, $i = 0, 1, \ldots$. Since $|A \cup C| = 2k-2$ and $n \ge 4k^2$, so we have at least $2k$ segments and at most $2k-2$ segments intersecting $A \cup C$. Hence there is a $L'$ avoiding $A \cup C$. $\square$