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Inclusion of forbidden minors in random representable matroids

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ABSTRACT

In 1984, Kelly and Oxley introduced the model of a random representable matroid $M[A_n]$ corresponding to a random matrix $A_n \in \mathbb{F}_q^{m(n)\times n}$, whose entries are drawn independently and uniformly from $\mathbb{F}_q$. Whereas properties such as rank, connectivity, and circuit size have been well-studied, forbidden minors have not yet been analyzed. Here, we investigate the asymptotic probability as $n \to \infty$ that a fixed $\mathbb{F}_q$-representable matroid M is a minor of $M[A_n]$. (We always assume $m(n) \geq \operatorname{rank}(M)$ for all sufficiently large n, otherwise M can never be a minor of the corresponding $M[A_n]$.) When M is free, we show that M is asymptotically almost surely (a.a.s.) a minor of $M[A_n]$. When M is not free, we show a phase transition: M is a.a.s. a minor if $n - m(n) \to \infty$, but is a.a.s. not if $m(n) - n \to \infty$. In the more general settings of $m \leq n$ and $m > n$, we give lower and upper bounds, respectively, on both the asymptotic and non-asymptotic probabilities that M is a minor of $M[A_n]$. The tools we develop to analyze matroid operations and minors of random matroids may be of independent interest.

Our results directly imply that $M[A_n]$ is a.a.s. not contained in any proper, minor-closed class $\mathcal{M}$ of $\mathbb{F}_q$-representable matroids, provided: (i) $n - m(n) \to \infty$, and (ii) $m(n)$ is at least the minimum rank of any $\mathbb{F}_q$-representable forbidden minor of $\mathcal{M}$, for all sufficiently large n. As an application, this shows that graphic matroids are a vanishing subset of linear matroids, in a sense made precise in the paper. Our results provide an approach for applying the rich theory around matroid minors to the less-studied field of random matroids.

1. Introduction

The motivation of this paper is to connect the study of random matroids with the rich theory recently developed around matroid minors. We ask a natural question: when does a fixed minor occur in the column dependence matroid obtained from a random matrix?

Formally, we consider Kelly and Oxley's model of a random representable matroid $M[A_n]$ corresponding to a random matrix $A_n \in \mathbb{F}_q^{m(n)\times n}$, whose entries are drawn independently and uniformly from the Galois field $\mathbb{F}_q$ of order q [9]. We denote this uniform distribution over random matrices in $\mathbb{F}_q^{m(n)\times n}$ succinctly by $[U_q]^{m(n)\times n}$, and write $A_n \sim [U_q]^{m(n)\times n}$ to indicate that A_n is drawn according to it. This paper investigates the asymptotic probability as $n \to \infty$ that a fixed $\mathbb{F}_q$-representable matroid M is a minor of $M[A_n]$.

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Interestingly, we are able to characterize the asymptotic probability that M is a minor of $M[A_n]$ solely by how fast the number of rows $m(n)$ of A_n grows. Observe that $M[A_n]$ can *never* have M as a minor if $m(n)$ is less than the rank $r(M)$ of M. Thus, throughout the paper, we impose the constraint that $m(n) \geq r(M)$ for all sufficiently large n.

We first show that every fixed *free* matroid M is asymptotically almost surely (a.a.s.) a minor of $M[A_n]$. We also give a closed-form expression for the non-asymptotic probability that this occurs, in terms of Gaussian coefficients.

However, inclusion of *non-free* minors is not as simple. Formally, for any finite field $\mathbb{F}_q$ and any non-free, $\mathbb{F}_q$-representable matroid M, we show that the following phase transition occurs:

$$\lim_{n\to\infty} \mathbb{P}_{A_n \sim [U_q]^{m(n)\times n}} \{M \text{ is a minor of } M[A_n]\} = \begin{cases} 1 & \text{if } n - m(n) \to \infty \\ 0 & \text{if } m(n) - n \to \infty. \end{cases}$$

Along the way, we show that $M[A_n]$ is a.a.s. the free matroid on n elements when $m(n) - n \to \infty$, extending a result of [9].

We also analyze the threshold between $n - m(n) \to \infty$ and $m(n) - n \to \infty$. As will be discussed formally later–but can be seen intuitively from the above phase transition–whether $m(n)$ is smaller or larger than n results in very different behaviors. So we investigate two cases: either $m(n) \geq n$ for all sufficiently large n, or $m(n) < n$ for all sufficiently large n.

In the case that $m(n) \geq n$ for all sufficiently large n, we show that for any non-free, $\mathbb{F}_q$-representable matroid M:

$$\limsup_{n\to\infty} \mathbb{P}_{A_n \sim [U_q]^{m(n)\times n}} \{M \text{ is a minor of } M[A_n]\} \leq 1 - C_q$$

where $C_q = \prod_{k=1}^{\infty} \left(1 - q^{-k}\right)$, and the limit superior is used only because the limit might not exist. (In the main text, we give intuition for the constant $C_q > 0$ by equating it to the asymptotic probability that a square matrix in $\mathbb{F}_q^{n\times n}$ is invertible [4].) In order to prove this asymptotic bound, we show the following non-asymptotic bound that holds for any $m(n) \geq n$:

$$\mathbb{P}_{A \sim [U_q]^{m(n)\times n}} \{M \text{ is a minor of } M[A]\} \leq 1 - \prod_{i=0}^{n-1} (1 - q^{i-m(n)}).$$

Next, in the case that $m(n) < n$ for all sufficiently large n, we show that provided[2] $m(n) \geq |E|$ for all sufficiently large n, then for any non-free, $\mathbb{F}_q$-representable matroid $M = (E, I)$ with ℓ loops:

$$\liminf_{n\to\infty} \mathbb{P}_{A_n \sim [U_q]^{m(n)\times n}} \{M \text{ is a minor of } M[A_n]\} > \left(1 - q^{-|E|}\right) p_{|E|-1,q,M}$$

where $p_{s,q,M} \in (0, 1)$ is defined as:

$$p_{s,q,M} = \binom{|E|}{\ell} \left(\frac{(q-1)^{|E|-r(M)-\ell}}{q^{s(|E|-r(M))}} \right) \prod_{i=0}^{r(M)-1} \left(1 - q^{i-s}\right).$$

Again, the limit inferior is used only because the limit might not exist. In order to prove this asymptotic bound, we show the following non-asymptotic bound that holds for any $m \geq r(M)$ and $n \geq |E|$:

$$\mathbb{P}_{A \sim [U_q]^{m\times n}} \{M \text{ is a minor of } M[A]\} > \max_{k \in \mathbb{Z}_+,\ k \leq \min(n-|E|, m-r(M))} \left(1 - q^{-(n-k)}\right) \left(1 - \left(1 - p_{m-k,q,M}\right)^{\lfloor \frac{n-k}{|E|} \rfloor}\right).$$

We note that this second setting $m(n) < n$ is significantly more involved because then $A_n \in \mathbb{F}_q^{m(n)\times n}$ is guaranteed to have dependence relations between the columns. Intuitively, this means that we will likely require contractions (in addition to just deletions) to obtain M as a minor of $M[A_n]$. But it is not even immediately clear how we should take contractions on a random matrix. The machinery we develop in order to achieve this may be of independent interest (see Section 4.3 for an overview of these tools).

Our final result allows us to leverage the connection between matroid characterizations and forbidden minors. Specifically, we show how our above results imply that, as $n \to \infty$, the random matroid $M[A_n]$ is a.a.s. not in *any* fixed proper, minor-closed class $\mathcal{M}$ of $\mathbb{F}_q$-representable matroids, provided: (i) $n - m(n) \to \infty$, and (ii) $m(n)$ is at least the minimum rank of the $\mathbb{F}_q$-representable forbidden minors of $\mathcal{M}$, for all sufficiently large n. As an example application, this directly shows that graphic matroids are a vanishing subset of linear matroids, with respect to the uniform random distribution $[U_q]^{m(n)\times n}$, and under mild constraints on the number of rows $m(n)$.

Along the way, we establish various results about random representable matroids and the uniform distribution $[U_q]^{m\times n}$. We note that our techniques rely heavily on properties of the uniform distribution $[U_q]^{m\times n}$, so generalizing to other distributions over $\mathbb{F}_q^{m\times n}$ would likely require new machinery.

1.1. Related work

Random matrices and especially random graphs have become increasingly well understood [2,3,6,19,20], but the field of random matroid theory is still much less explored. The works of [7,8,11–14,17] analyze $\mathbb{F}_q$-representable random

[2] There is an analogue that only requires $m(n) \geq r(M)$ for sufficiently large n, but the resulting bound is messier.