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Note

Maximal k-edge-colorable subgraphs, Vizing's Theorem, and Tuza's Conjecture

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ABSTRACT

We prove that if M is a maximal k-edge-colorable subgraph of a multigraph G and if $F = \{v \in V(G) : d_M(v) \leq k - \mu(v)\}$, then $d_F(v) \leq d_M(v)$ for all $v \in V(G)$ with $d_M(v) < k$. (When G is a simple graph, the set F is just the set of vertices having degree less than k in M.) This implies Vizing's Theorem as well as a special case of Tuza's Conjecture on packing and covering of triangles. A more detailed version of our result also implies Vizing's Adjacency Lemma for simple graphs.

1. Introduction

A *proper k-edge-coloring* of a multigraph G without loops is a function $\psi : E(G) \to [k]$ such that $\psi(e) \neq \psi(f)$ whenever e and f are distinct edges sharing an endpoint (or both endpoints), where $[k] = \{1, \ldots, k\}$. A graph is *k-edge-colorable* if it admits a proper k-edge-coloring. We will tacitly assume in the rest of this paper that all multigraphs under consideration are loopless.

A fundamental theorem concerning edge-coloring is Vizing's Theorem [30]. Given a multigraph G, we write $\mu_G(v, w)$ for the number of edges joining two vertices v and w, and we write $\mu_G(v)$ for $\max_{w \in V(G)} \mu_G(v, w)$. When the graph G is understood, we omit the subscripts. We also write $\Delta(G)$ for the maximum degree of G and $\mu(G)$ for $\max_{v \in V(G)} \mu(v)$. Vizing's Theorem can then be stated as follows:

Theorem 1.1 (*Vizing [30]*)**.** *If G is a multigraph and $k \geq \Delta(G) + \mu(G)$, then G is k-edge-colorable.*

Following the notation of [27], let $\Delta^{\mu}(G) = \max_{v \in V(G)}[d(v) + \mu(v)]$. Since $\Delta^{\mu}(G) \leq \Delta(G) + \mu(G)$ for any multigraph G, and since this inequality is sometimes strict, the following theorem of Ore [22] strengthens Theorem 1.1.

Theorem 1.2 (*Ore [22]*)**.** *If G is a multigraph and $k \geq \Delta^{\mu}(G)$, then G is k-edge-colorable.*

In this paper, we prove the following generalization of Theorem 1.2. Here, when $F \subset V(G)$, we write $d_F(v)$ for $\sum_{w \in F} \mu(v, w)$, and when $M \subset E(G)$, we write $d_M(v)$ for the total number of M-edges incident to v.

Theorem 1.3. *Let G be a multigraph, let $k \geq 1$, and let M be a maximal k-edge-colorable subgraph of G. If $F = \{v \in V(G) : d_M(v) \leq k - \mu(v)\}$, then for every $v \in V(G)$ with $d_M(v) < k$, we have $d_F(v) \leq d_M(v)$.*

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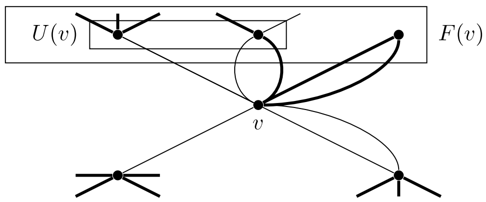

Fig. 1. Illustration of $F(v)$ and $U(v)$ for a vertex v, in the case $k = 4$. Thick edges denote edges in M; vertices have no incident edges aside from those pictured.

Theorem 1.3 is easiest to understand in the case of simple graphs, where $\mu(v) = 1$ for all v. In this case, F is just the set of all vertices with fewer than k colors present on the incident edges, that is, the set of all vertices missing at least one color.[2]

It is also instructive to consider Theorem 1.3 in the cases $k = 1$ and $k = 2$. Since a maximal matching in a graph G is just a maximal 1-edge-colorable subgraph of G, the $k = 1$ case of Theorem 1.3 just states the observation that the set of vertices left uncovered by a maximal matching is independent.

In the case $k = 2$, we can observe that in a maximal 2-edge-colorable subgraph $M \subset G$, every component of M is an even cycle or a path (possibly a 1-vertex path), and the vertices of F are the endpoints of the path components. Theorem 1.3 then states that $G[F]$ induces a graph consisting of a matching together with possibly some isolates, where all vertices isolated in M are also isolated in $G[F]$. This conclusion is not difficult to prove directly, as the maximality of M implies that the only G-edges among the vertices of F are edges that join the endpoints of the same path, if this would yield an odd cycle.

For $k > 2$, no simple characterization of k-edge colorable graphs is known, so a direct appeal to the structure of M is not possible. However, Theorem 1.3 still yields the following corollary.

Corollary 1.4. *If G is a simple graph, M is a maximal k-edge-colorable subgraph of G, and F is the set of vertices with fewer than k incident M-edges, then $\Delta(G[F]) \leq k - 1$.*

To see that Theorem 1.3 implies Theorem 1.2, observe that if $k \geq \Delta^{\mu}(G)$ and M is a maximal k-edge-colorable subgraph of G, then $F = V(G)$, so Theorem 1.3 states that $d_M(v) \geq d_G(v)$ for every vertex v. As M is a subgraph of G, this implies $M = G$, so that G is k-edge-colorable. In Section 3, we show that Theorem 1.3 also implies a multigraph version of a strengthening of Vizing's Theorem due to Lovász and Plummer [20] and to Berge and Fournier [5].

In order to prove Theorem 1.3, we actually prove a more technical version of the theorem, with a somewhat stronger conclusion. This version of Theorem 1.3 is similar to Vizing's Adjacency Lemma, and we explore the connection in more detail in Section 5.

Definition 1.5. Given a multigraph G, a subgraph $M \subset G$, and an integer $k \geq 1$, for each $v \in V(G)$ we define vertex sets $F(v)$ and $U(v)$ by

$$F(v) = \{w \in N(v) : \ d_M(w) \leq k - \mu_G(v, w)\},$$
$$U(v) = \{w \in F(v) : \ \mu_M(v, w) < \mu_G(v, w)\}.$$

We also write $d^F(v)$ for $d_{F(v)}(v)$, that is, $d^F(v)$ is the total number of edges from v to the vertices in $F(v)$. The superscript here is meant to emphasize that the F in this notation is a set depending on v, rather than being a fixed set as in Theorem 1.3. Fig. 1 illustrates the definition of $F(v)$ and $U(v)$.

Theorem 1.6. *Let G be a multigraph, let $k \geq 1$, and let M be a maximal k-edge-colorable subgraph of G. For every $v \in V(G)$ with $d_M(v) < k$, we have*

$$d^F(v) \leq d_M(v) - \sum_{w \in U(v)} (k - d_M(w) - \mu_G(v, w)).$$

Note that since $U(v) \subset F(v)$ by definition, we have $d_M(w) \leq k - \mu_G(v, w)$ for all $w \in U(v)$, so that each term $k - d_M(w) - \mu_G(v, w)$ in the above sum is nonnegative. Furthermore, when F_0 is the set defined in Theorem 1.3, we see that $(N(v) \cap F_0) \subset F(v)$ for all $v \in V(G)$. Thus, Theorem 1.6 indeed strengthens Theorem 1.3.

We now consider a conjecture of Tuza regarding packing and covering of triangles.

Definition 1.7. Given a graph G, let $\tau(G)$ denote the minimum size of an edge set X such that $G - X$ is triangle-free, and let $\nu(G)$ denote the maximum size of a set of pairwise edge-disjoint triangles in G.

2 The letter F is meant to evoke the word "deficient", the letter D being unavailable since it is used in a different context in this paper.