Discrete Mathematics 340 (2017) 1584–1597

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Identifying codes for infinite triangular grids with a finite number of rows

Rennan Dantas [a], Frédéric Havet [b,c], Rudini M. Sampaio [a,*]

[a] Universidade Federal do Ceará, Brazil
[b] Project COATI, I3S(CNRS & University of Nice Sophia Antipolis), France
[c] INRIA, France

ARTICLE INFO

Article history:
Received 2 June 2016
Received in revised form 21 December 2016
Accepted 17 February 2017

Keywords:
Identifying codes
Triangular grids

ABSTRACT

A set $C \subset V(G)$ is an *identifying code* in a graph G if for all $v \in V(G)$, $C[v] \neq \emptyset$, and for all distinct $u, v \in V(G)$, $C[u] \neq C[v]$, where $C[v] = N[v] \cap C$ and $N[v]$ denotes the closed neighborhood of v in G. The minimum density of an identifying code in G is denoted by $d^*(G)$. Given a positive integer k, let T_k be the triangular grid with k rows. In this paper, we prove that $d^*(T_1) = d^*(T_2) = 1/2$, $d^*(T_3) = d^*(T_4) = 1/3$, $d^*(T_5) = 3/10$, $d^*(T_6) = 1/3$ and $d^*(T_k) = 1/4 + 1/(4k)$ for every $k \geq 7$ odd. Moreover, we prove that $1/4 + 1/(4k) \leq d^*(T_k) \leq 1/4 + 1/(2k)$ for every $k \geq 8$ even. We conjecture that $d^*(T_k) = 1/4 + 1/(2k)$ for every $k \geq 8$ even.

1. Introduction

Let G be a graph G and v a vertex of G. The *neighborhood* of v, denoted by $N(v)$, is the set of vertices adjacent to v in G, and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$.

Given a set $C \subseteq V(G)$, the *identifier* of a vertex $v \in V(G)$ is $C[v] = N[v] \cap C$. We say that C is an *identifying code* of G if every vertex has non-empty identifier and, for all distinct u and v, u and v have distinct identifiers. Formally, C is an *identifying code* if

(i) for all $v \in V(G)$, $C[v] \neq \emptyset$, and
(ii) for all distinct $u, v \in V(G)$, $C[u] \neq C[v]$.

Hence an identifying code is a set such that the vertices have non-empty distinct identifiers.

Let G be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer r and vertex v, we denote by $B_r(v)$ the ball of radius r in G centered at v, that is $B_r(v) = \{x \mid \text{dist}(v, x) \leq r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of C in G, denoted by $d(C, G)$, is defined by

$$d(C, G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|},$$

where v_0 is an arbitrary vertex in G. The infimum of the density of an identifying code in G is denoted by $d^*(G)$. Observe that if G is finite, then $d^*(G) = |C^*|/|V(G)|$, where C^* is a minimum-size identifying code of G.

The problem of finding low-density identifying codes was introduced in [9] in relation to fault diagnosis in arrays of processors. Here the vertices of an identifying code correspond to controlling processors able to check themselves and their

* Corresponding author.
E-mail addresses: rennan@lia.ufc.br (R. Dantas), frederic.havet@cnrs.fr (F. Havet), rudini@lia.ufc.br (R.M. Sampaio).

http://dx.doi.org/10.1016/j.disc.2017.02.015

neighbors. Thus the identifying property guarantees location of a faulty processor from the set of "complaining" controllers. Identifying codes are also used in [10] to model a location detection problem with sensor networks.

Particular interest was dedicated to grids as many processor networks have a grid topology. There are three regular infinite grids in the plane, namely the hexagonal grid, the square grid and the triangular grid.

Regarding the infinite hexagonal grid $\mathcal{G}_H$, the best upper bound on $d^*(\mathcal{G}_H)$ is 3/7 and comes from two identifying codes constructed by Cohen et al. [5]; these authors also proved a lower bound of 16/39. This lower bound was improved to 12/29 by Cranston and Yu [6]. Cukierman and Yu [7] further improved it to 5/12.

Regarding the infinite square grid $\mathcal{G}_S$, Cohen et al. [3] gave a periodic identifying code of $\mathcal{G}_S$ with density 7/20. This density was later proved to be optimal by Ben-Haim and Litsyn [1]. Some papers also obtained results for square grids with finite number of rows. For any positive integer k, let S_k denote the square grid with k rows. Daniel, Gravier, and Moncel [8] showed that $d^*(S_1) = 1/2$ and $d^*(S_2) = 3/7$. They also showed that, for every $k \geq 3$, $\frac{7}{20} - \frac{1}{2k} \leq d^*(S_k) \leq \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{2}{k}\right\}$. These bounds were recently improved by Bouznif et al. [2] who established

$$\frac{7}{20} + \frac{1}{20k} \leq d^*(S_k) \leq \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{3}{10k}\right\}.$$

They also proved $d^*(S_3) = 3/7$.

The *infinite triangular grid* $\mathcal{G}_T$ is the infinite graph with vertices in $\mathbb{Z} \times \mathbb{Z}$ such that $N((x, y)) = \{(x, y \pm 1), (x \pm 1, y), (x - 1, y + 1), (x + 1, y - 1)\}$. Given an integer $k \geq 2$, let $[k] = \{1, \ldots, k\}$ and let T_k be the subgraph of $\mathcal{G}_T$ induced by the vertex set $\{(x, y) \in \mathbb{Z} \times [k]\}$. Karpovsky et al. [9] showed that $d^*(\mathcal{G}_T) = 1/4$. Trivially, $T_1 = S_1$. Hence $d^*(T_1) = 1/2$. In this paper, we prove several results regarding the density of an identifying code of T_k, $k > 1$. We prove that $d^*(T_2) = 1/2$, $d^*(T_3) = d^*(T_4) = 1/3$, $d^*(T_5) = 3/10$, $d^*(T_6) = 1/3$ and $d^*(T_k) = 1/4 + 1/(4k)$ for every $k \geq 7$ odd. Moreover, we prove that $1/4 + 1/(4k) \leq d^*(T_k) \leq 1/4 + 1/(2k)$ for every $k \geq 8$ even. We conjecture that the upper bound is the optimum density.

2. The infinite triangular grid with two, three or six rows

In this section we prove the following theorem.

Theorem 1. *$d^*(T_2) = 1/2$, $d^*(T_3) = 1/3$ and $d^*(T_6) = 1/3$.*

From the identifying codes of Figs. 1–3, we obtain the following upper bounds: $d^*(T_2) \leq 1/2$, $d^*(T_3) \leq 1/3$ and $d^*(T_6) \leq 1/3$. In other words, consider the sets $C_{2,a}, C_{2,b}, C_{3,a}, C_{3,b}, C_6$ given below.

$$\begin{aligned}
C_{2,a} &= \{(x, 1) \mid x \equiv 1, 3 \bmod 5\} \cup \{(x, 2) \mid x \equiv 1, 2, 4 \bmod 5\};\\
C_{2,b} &= \{(x, 1) \mid x \equiv 1, 2, 3, 4 \bmod 5\} \cup \{(x, 2) \mid x \equiv 2 \bmod 5\};\\
C_{3,a} &= \{(x, 1) \mid x \equiv 1 \bmod 2\} \cup \{(x, 3) \mid x \equiv 1 \bmod 2\};\\
C_{3,b} &= \{(x, 1) \mid x \equiv 1 \bmod 3\} \cup \{(x, 2) \mid x \equiv 2 \bmod 3\} \cup \{(x, 3) \mid x \equiv 3 \bmod 3\};\\
C_6 &= \{(x, 1), (x, 3) \mid x \text{ odd }\} \cup \{(x, 5) \mid x \in \mathbb{Z}\}.
\end{aligned}$$

It is easy to check that $C_{2,a}$ and $C_{2,b}$ are identifying codes of T_2 with density 1/2, $C_{3,a}$ and $C_{3,b}$ are identifying codes of T_3 with density 1/3, and C_6 is an identifying code of T_6 with density 1/3.

In order to prove the lower bounds, we introduce the notion of quasi-identifying code of T_3. Roughly speaking, a quasi-identifying code of T_3 does not care about the last row. Formally, a *quasi-identifying code* C of T_3 is a subset $C \subseteq V(T_3)$ such that

(i) for all $v \in V(T_2)$, $C[v] \neq \emptyset$, and
(ii) for all distinct $u, v \in V(T_2)$, $C[u] \neq C[v]$.

The main technical result of this section is the following.

Lemma 2. *Every quasi-identifying code C' of T_3 has density $d(C', T_3)$ at least* 1/3.

Before proving this lemma, we show that this result implies Theorem 1.

Proof of Theorem 1. Notice that every identifying code C'_3 of T_3 is obviously a quasi-identifying code of T_3. Then, from Lemma 2, $d(C'_3, T_3) \geq 1/3$.

Also notice that every identifying code C'_2 of T_2 is also a quasi-identifying code of T_3. Then, from Lemma 2, $d(C'_2, T_3) \geq 1/3$. Since $d(C'_2, T_3) = 2 \cdot d(C'_2, T_2)/3$, we obtain $d(C'_2, T_2) \geq 1/2$.

Finally notice that every identifying code C'_6 of T_6 induces the following two quasi-identifying codes of T_3:

$$\begin{aligned}
C'_{3,a} &= \{(x, y) \mid (x, y) \in C'_6 \text{ and } y \in \{1, 2, 3\}\};\\
C'_{3,b} &= \{(x, 7 - y) \mid (x, y) \in C'_6 \text{ and } y \in \{4, 5, 6\}\}.
\end{aligned}$$

Then $d(C'_6, T_6) \geq 1/3$, since, from Lemma 2, $d(C'_{3,a}, T_3) \geq 1/3$ and $d(C'_{3,b}, T_3) \geq 1/3$. □