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Low minor 5-stars in 3-polytopes with minimum degree 5 and no 6-vertices[☆]

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ABSTRACT

In 1940, in attempts to solve the Four Color Problem, H. Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class P_5 of 3-polytopes with minimum degree 5.

Given a 3-polytope P, by $h(P)$ we denote the minimum of the maximum degrees (height) of the neighborhoods of 5-vertices (minor 5-stars) in P.

In 1996, S. Jendrol' and T. Madaras showed that if a polytope P in P_5 is allowed to have a 5-vertex adjacent to four 5-vertices, then $h(P)$ can be arbitrarily large.

For each P without vertices of degree 6 and 5-vertices adjacent to four 5-vertices in P_5, it follows from Lebesgue's Theorem that $h(P) \leq 41$. Recently, this bound was lowered to $h(P) \leq 28$ by O. Borodin, A. Ivanova, and T. Jensen and then to $h(P) \leq 23$ by O. Borodin and A. Ivanova.

In this paper, we prove that every such polytope P satisfies $h(P) \leq 17$, which bound is sharp.

1. Introduction

The degree of a vertex or face x in a convex finite 3-dimensional polytope (called a *3-polytope*) is denoted by $d(x)$. A *k-vertex* is a vertex v with $d(v) = k$. A *k^+-vertex* (*k^--vertex*) is one of degree at least k (at most k). Similar notation is used for the faces. A 3-polytope with minimum degree δ is denoted by P_δ. The *weight* of a subgraph S of a 3-polytope is the sum of degrees of the vertices of S in the 3-polytope. The *height* of a subgraph S of a 3-polytope is the maximum degree of the vertices of S in the 3-polytope. A k-star, a star with k rays, $S_k(v)$ is *minor* if its center v has degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor. By $w(S_k)$ and $h(S_k)$ we denote the minimum weight and height, respectively, of minor k-stars in a given 3-polytope.

In 1904, Wernicke [15] proved that every P_5 has a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [9] in 1922 to the existence of a 5-vertex with two 6^--neighbors. In 1940, in attempts to solve the Four Color Problem, Lebesgue [13, p. 36] gave an approximate description of the neighborhoods of 5-vertices in P_5's. In particular, this description implies the results in [9,15] and shows that there is a 5-vertex with three 7^--neighbors.

For P_5's, the bounds $w(S_1) \leq 11$ (Wernicke [15]) and $w(S_2) \leq 17$ (Franklin [9]) are tight. It was proved by Lebesgue [13] that $w(S_3) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [11] to the sharp bound $w(S_3) \leq 23$. Furthermore, Jendrol' and Madaras [11] gave a precise description of minor 3-stars in P_5's. Lebesgue [13] proved $w(S_4) \leq 31$, which was strengthened by Borodin and Woodall [8] to the tight bound $w(S_4) \leq 30$. Note that $w(S_3) \leq 23$ easily implies $w(S_2) \leq 17$ and immediately follows from $w(S_4) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of the minimum weight). Recently, Borodin and Ivanova obtained a precise description of 4-stars in P_5's [4].

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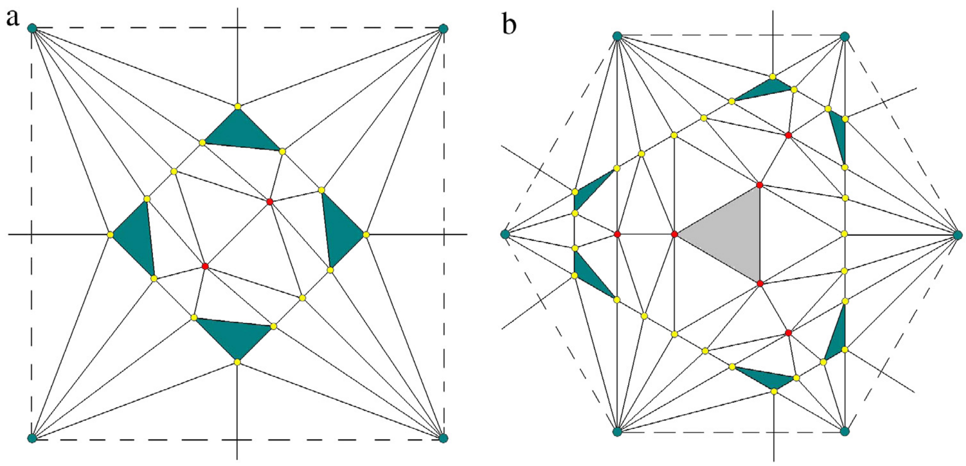

Fig. 1. Replacements for faces in Fig. 2: (a) B_4 for each 4-face and (b) B_6 for each 6-face.

For arbitrary 3-polytopes, that is for P_3's, the following results concerning $(d-2)$-stars at d-vertices, $d \le 5$, are known. Van den Heuvel and McGuinness [14] proved (in particular) that either $w(S_1(v)) \le 14$ with $d(v) = 3$, or $w(S_2(v)) \le 22$ with $d(v) = 4$, or $w(S_3(v)) \le 29$ with $d(v) = 5$. Balogh et al. [1] proved that there is a 5^--vertex adjacent to at most two 11^+-vertices. Harant and Jendrol' [10] strengthened these results by proving (in particular) that either $w(S_1(v)) \le 13$ with $d(v) = 3$, or $w(S_2(v)) \le 19$ with $d(v) = 4$, or $w(S_3(v)) \le 23$ with $d(v) = 5$. Recently, we obtained a precise description of $(d-2)$-stars in P_3's [5].

For P_3's, the problem of describing $(d-1)$-stars at d-vertices, $d \le 5$, called *pre-complete stars*, appears difficult. As follows from the double n-pyramid, the minimum weight $w(S_{d-1})$ of pre-complete stars in P_4's can be arbitrarily large. Even when $w(S_{d-1})$ is restricted by appropriate conditions, the tight upper bounds on it are unknown. Borodin et al. [2,3] proved (in particular) that if a planar graph with $\delta \ge 3$ has no edge joining two 4^--vertices, then there is a star $S_{d-1}(v)$ with $w(S_{d-1}(v)) \le 38 + d(v)$, where $d(v) \le 5$ (see [2, Theorem 2.A]). Jendrol' and Madaras [12] proved that if the weight $w(S_1)$ of every edge in an P_3 is at least 9, then there is a pre-complete star of height at most 20, where the bound of 20 is best possible.

The more general problem of describing d-stars at d-vertices, $d \le 5$, called *complete stars*, at the moment seems untractable for arbitrary 3-polytopes and difficult even for P_5s.

Here, we consider the 3-polytopes with $\delta = 5$ and no 6-vertices. By $h(P)$ we denote the minimum height of minor 5-stars in 3-polytope P. Jendrol' and Madaras [11] showed that if a polytope P_5 has a 5-vertex adjacent to four 5-vertices, then $h(P_5)$ can be arbitrarily large.

For each P_5 that has neither 6-vertices nor 5-vertices adjacent to four 5-vertices, it follows from Lebesgue's Theorem that $h(P_5) \le 41$. Recently, this bound was lowered to $h(P_5) \le 28$ by Borodin, Ivanova, and Jensen [7] and then to $h(P_5) \le 23$ in Borodin–Ivanova [6].

The purpose of this note is to prove the following fact.

Theorem 1. *Every 3-polytope P with minimum degree 5 and neither 6-vertices nor 5-vertices adjacent to four 5-vertices satisfies $h(P) \le 17$, which bound is best possible.*

As shown in [7], there is a 3-polytope P with minimum degree 5 and no 5-vertices adjacent to four 5-vertices such that $h(P) = 20$.

2. Proof of Theorem 1

The tightness of the bound 17

In Fig. 1(a), we see a configuration B_4 to be put into each 4-face f of the graph in Fig. 2 so as the two opposite vertices in f labeled with 5 correspond to the boundary vertices in B_4 having five ingoing edges. Note that the other two boundary vertices of f have four ingoing edges each.

The configuration B_6 in Fig. 1(b) has six boundary vertices, whose ingoing degrees alternate from 4 to 7. We put a copy of B_6 into each of the four 6-faces in Fig. 2 according to the labeling.

After inserting all B_4s and B_6s into the graph H in Fig. 2 followed by deleting all original edges in H, we obtain a semi-triangulation. Gluing two such semi-triangulations along the outside cycle results in a triangulation with $\delta = 5$ and such that each 5-vertex is adjacent to a vertex of degree at least $7+5+5 = 5+4+4+4 = 17$ and another 7^+-vertex, as desired. This construction has about 3000 vertices.