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Proof of a refinement of Blum's conjecture on hexagonal dungeons

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ABSTRACT

Matt Blum conjectured that the- number of tilings of a hexagonal dungeon with side-lengths $a, 2a, b, a, 2a, b$ (for $b \geq 2a$) equals $13^{2a^2}14^{\lfloor a^2/2 \rfloor}$. Ciucu and the author of the present paper proved the conjecture by using Kuo's graphical condensation method. In this paper, we investigate a 3-parameter refinement of the conjecture and its application to enumeration of tilings of several new types of the hexagonal dungeons.

1. Introduction

A lattice partitions the plane into fundamental regions. A *(lattice) region* considered in this paper is a finite connected union of fundamental regions. We call the union of any two fundamental regions sharing an edge a *tile*. We would like to know how many different ways there are to cover a region by tiles so that there are no gaps or overlaps; and such coverings are called *tilings*. We use the notation $\mathrm{M}(R)$ for the number of tilings of a region R, and $\mathcal{M}(R)$ for the set of all tilings of R.

The study of enumeration of tilings and perfect matchings dates back to the early 1900s when P. A. MacMahon [22] proved his classical theorem on plane partitions fitting in a given box. His result yields a tiling enumeration for a centrally symmetric hexagon on the triangular lattice. In 1999, J. Propp published his well-known survey paper about the enumeration of tilings and perfect matchings [24] in which he listed 32 open problems in the field. One of the most challenging problems on the list is the following conjecture of Matt Blum (listed as Problem 25 in [24]).

Consider the lattice obtained from the triangular lattice by drawing in all attitudes of each unit triangle. The resulting lattice is usually called the G_2-lattice, since it is the lattice corresponding to the affine Coxeter group G_2. On the G_2-lattice, Blum investigated a variation of Aztec dungeon (see [1]) called the *hexagonal dungeon*. In particular, we draw a hexagonal contour of side-lengths[1] $a, 2a, b, a, 2a, b$ (in cyclic order, start on the west side) like the light bold contour in Fig. 1.1. We draw next a jagged boundary running along the hexagonal contour (see the dark bold closed path in Fig. 1.1), and denote by $HD_{a,2a,b}$ the region restricted by the boundary. Blum found a striking pattern in the numbers of tilings of the hexagonal dungeons, which led him to his well-known conjecture that the hexagonal dungeon $HD_{a,2a,b}$ has $13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}$ tilings, when $b \geq 2a$. Fourteen years later, Ciucu and the author proved the conjecture in [6] by using Kuo's graphical condensation method [12]. However, the proof did not explain the (surprising) appearance of the numbers 13 and 14 in Blum's formula. In this paper, we consider the generating function of the tilings of the hexagonal dungeons and give an explanation for the appearance of the numbers 13 and 14.

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[1] The unit here is the side-length of the unit triangles.

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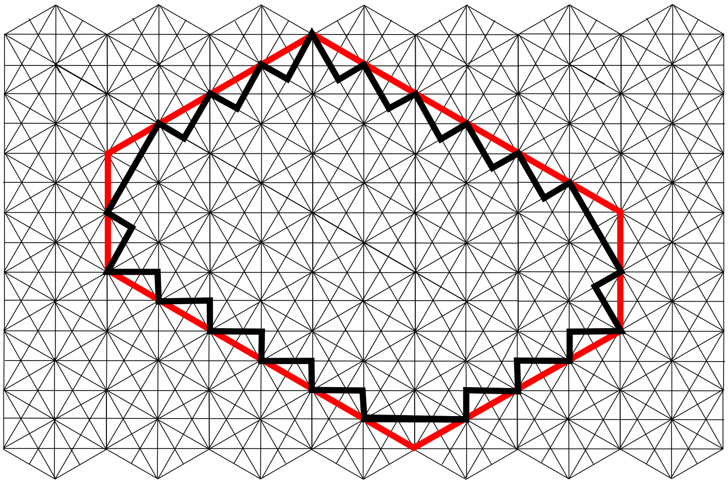

Fig. 1.1. The hexagonal dungeon of sides 2, 4, 6, 2, 4, 6 (in cyclic order, starting from the western side). This figure first appeared in [6].

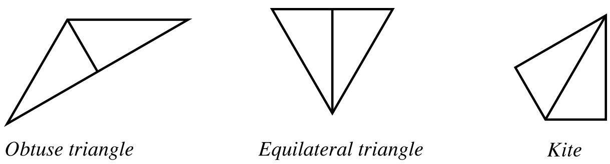

Fig. 1.2. Three types of tiles of a hexagonal dungeon.

The tiles in a hexagonal dungeon have three possible shapes: an obtuse triangle, an equilateral triangle, and a kite (see Fig. 1.2). We consider the following generating functions

$$F(x,y,z)=\sum_{T\in\mathcal{M}(HD_{a,2a,b})}x^m y^n z^l, \tag{1.1}$$

where m, n, l are respectively the numbers of obtuse triangle tiles, equilateral triangle tiles, and kite tiles in the tiling T of $HD_{a,2a,b}$. We call $F(x,y,z)$ the *tiling generating function* of the hexagonal dungeon. Our goal is to prove the following 3-parameter refinement of Blum's conjecture.

Theorem 1.1 (*Weighted Hexagonal Dungeon Theorem*)**.** *Assume a and b are two positive integers so that $b\geq 2a$. Then the tiling generating function of the hexagonal dungeon $HD_{a,2a,b}$ is given by*

$$\begin{aligned}F(x,y,z)&=\gamma(a)x^{3ab-2a^2+3\lfloor\frac{a^2}{2}\rfloor}y^{6ab-a^2+3a-b-2\lfloor\frac{a^2}{2}\rfloor}z^{9ab+3a^2-7\lfloor\frac{a^2}{2}\rfloor-2b-12a}\\&\quad\times(x^6+3x^4y^2+3x^2y^4+y^6+2x^3z^3+2xy^2z^3+z^6)^{2a^2}\\&\quad\times\left((x^4+2x^2y^2+y^4+x^3z+xy^2z-y^2z^2+xz^3+z^4)(x^2+y^2-xz+z^2)\right)^{\lfloor\frac{a^2}{2}\rfloor},\end{aligned} \tag{1.2}$$

where $\gamma(a)=(y/z)^{1-(-1)^a}$ (i.e. $\gamma(a)$ is 1 if a is even, and y^2/z^2 if a is odd).

The main ingredient of our proof is Kuo's *graphical condensation method*, a combinatorial version of the well-known *Dodgson condensation* (based on the *Jacobi–Desnannot identity*, see e.g. [7] and [23], pp. 136–149) in linear algebra. This method was first introduced by Eric H. Kuo [12] to (re)prove the Aztec diamond theorem by Elkies, Kuperberg, Larsen, and Propp [8,9]. Kuo condensation has become a power tool in the enumeration of tilings and perfect matchings. We refer the reader to e.g. [2,10,13,28–30] for various aspects and generalizations of the method; and e.g. [3–6,11,15–21,25–27,31] for recent applications of Kuo condensation.

The proof of Theorem 1.1 follows the lines in the proof of Blum's conjecture in [6]. In particular, we will show that the tiling generating function $F(x,y,z)$ of the hexagonal dungeon $HD_{a,2a,b}$ can be written in terms of the tiling generating function of a trapezoidal subregion. Next, we will use Kuo condensation to prove a simple product formula for the tiling generating function of the latter trapezoidal subregion. We will actually obtain a stronger result by showing an explicit formula for the tiling generating functions of two larger families of regions that contain the trapezoidal subregion (see Theorem 2.1).

The rest of this paper is organized as follows. In Section 2, we recall the definition of two important regions $D_{a,b,c}$ and $E_{a,b,c}$, which were first introduced in [6]. In addition, we state a result on their tiling generating functions (see Theorem 2.1), which