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Double-critical graph conjecture for claw-free graphs

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ABSTRACT

A connected graph G with chromatic number t is *double-critical* if $G \setminus \{x, y\}$ is $(t-2)$-colorable for each edge $xy \in E(G)$. The complete graphs are the only known examples of double-critical graphs. A long-standing conjecture of Erdős and Lovász from 1966, which is referred to as the *Double-Critical Graph Conjecture*, states that there are no other double-critical graphs. That is, if a graph G with chromatic number t is double-critical, then G is the complete graph on t vertices. This has been verified for $t \le 5$, but remains open for $t \ge 6$. In this paper, we first prove that if G is a non-complete, double-critical graph with chromatic number $t \ge 6$, then no vertex of degree $t+1$ is adjacent to a vertex of degree $t+1$, $t+2$, or $t+3$ in G. We then use this result to show that the Double-Critical Graph Conjecture is true for double-critical graphs G with chromatic number $t \le 8$ if G is claw-free.

1. Introduction

All graphs considered in this paper are finite and without loops or multiple edges. For a graph G, we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $e(G)$ the number of edges, $\alpha(G)$ the independence number, $\omega(G)$ the clique number, $\chi(G)$ the chromatic number, and $\overline{G}$ the complement of G. For a vertex $x \in V(G)$, we will use $N_G(x)$ to denote the set of vertices in G which are adjacent to x. We define $N_G[x] = N_G(x) \cup \{x\}$ and $d_G(x) = |N_G(x)|$. Given vertex sets $A, B \subseteq V(G)$, we say that A is *complete to* (resp. *anti-complete to*) B if for every $a \in A$ and every $b \in B$, $ab \in E(G)$ (resp. $ab \notin E(G)$). The subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \setminus A$ the set $B - A$, $e_G(A, B)$ the number of edges between A and B in G, and $G \setminus A$ the subgraph of G induced on $V(G) \setminus A$, respectively. If $A = \{a\}$, we simply write $B \setminus a$, $e_G(a, B)$, and $G \setminus a$, respectively. A graph H is an *induced subgraph* of a graph G if $V(H) \subseteq V(G)$ and $H = G[V(H)]$. A graph G is *claw-free* if G does not contain $K_{1,3}$ as an induced subgraph. Given two graphs G and H, the *union* of G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. A cycle with $t \ge 3$ vertices is denoted by C_t. Throughout this paper, a proper vertex coloring of a graph G with k colors is called a *k-coloring* of G.

In 1966, the following conjecture of Lovász was published by Erdős [6] and is known as the Erdős–Lovász Tihany Conjecture.

Conjecture 1.1. *For any integers $s, t \ge 2$ and any graph G with $\omega(G) < \chi(G) = s + t - 1$, there exist disjoint subgraphs G_1 and G_2 of G such that $\chi(G_1) \ge s$ and $\chi(G_2) \ge t$.*

To date, Conjecture 1.1 has been shown to be true only for values of $(s, t) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$. The case $(2, 2)$ is trivial. The case $(2, 3)$ was shown by Brown and Jung in 1969 [3]. Mozhan [10] and Stiebitz [14] each independently showed the case $(2, 4)$ in 1987. The cases $(3, 3)$, $(3, 4)$, and $(3, 5)$ were also settled by Stiebitz in 1987 [15].

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Recent work on the Erdős–Lovász Tihany Conjecture has focused on proving the conjecture for certain classes of graphs. Kostochka and Stiebitz [9] showed the conjecture holds for line graphs. Balogh, Kostochka, Prince, and Stiebitz [2] then showed that the conjecture holds for all quasi-line graphs and all graphs G with $\alpha(G) = 2$. More recently, Chudnovsky, Fradkin, and Plumettaz [4] proved the following slight weakening of Conjecture 1.1 for claw-free graphs, the proof of which is long and relies heavily on the structure theorem for claw-free graphs developed by Chudnovsky and Seymour [5].

Theorem 1.2. *Let G be a claw-free graph with $\chi(G) > \omega(G)$. Then there exists a clique K with $|V(K)| \leq 5$ such that $\chi(G \setminus V(K)) > \chi(G) - |V(K)|$.*

The most recent result related to the Erdős–Lovász Tihany Conjecture is due to Stiebitz [13], who showed that for integers $s, t \geq 2$, any graph G with $\omega(G) < \chi(G) = s + t - 1$ contains disjoint subgraphs G_1 and G_2 of G with either $\chi(G_1) \geq s$ and $\mathrm{col}(G_2) \geq t$, or $\mathrm{col}(G_1) \geq s$ and $\chi(G_2) \geq t$, where $\mathrm{col}(H)$ denotes the coloring number of a graph H.

If we restrict $s = 2$ in Conjecture 1.1, then the Erdős–Lovász Tihany Conjecture states that for any graph G with $\chi(G) > \omega(G) \geq 2$, there exists an edge $xy \in E(G)$ such that $\chi(G \setminus \{x, y\}) \geq \chi(G) - 1$. To prove this special case of Conjecture 1.1, suppose for a contradiction that no such edge exists. Then $\chi(G \setminus \{x, y\}) = \chi(G) - 2$ for every edge $xy \in E(G)$. This motivates the definition of double-critical graphs. A connected graph G is double-critical if for every edge $xy \in E(G)$, $\chi(G \setminus \{x, y\}) = \chi(G) - 2$. A graph G is *t-chromatic* if $\chi(G) = t$. We are now ready to state the following conjecture, which is referred to as the *Double-Critical Graph Conjecture*, due to Erdős and Lovász [6].

Conjecture 1.3. *Let G be a double-critical, t-chromatic graph. Then $G = K_t$.*

Since Conjecture 1.3 is a special case of Conjecture 1.1, it has been settled in the affirmative for $t \leq 5$ [10,14], for line graphs [9], and for quasi-line graphs and graphs with independence number two [2]. Representing a weakening of Conjecture 1.3, Kawarabayashi, Pedersen, and Toft [8] have shown that any double-critical, t-chromatic graph contains K_t as a minor for $t \in \{6, 7\}$. As a further weakening, Pedersen [11] showed that any double-critical, 8-chromatic graph contains K_8^- as a minor. Albar and Gonçalves [1] later proved that any double-critical, 8-chromatic graph contains K_8 as a minor. Their proof is computer-assisted. The present authors [12] gave a computer-free proof of the same result and further showed that any double-critical, t-chromatic graph contains K_9 as a minor for all $t \geq 9$. We note here that Theorem 1.2 does not completely settle Conjecture 1.3 for all claw-free graphs. Recently, Huang and Yu [7] proved that the only double-critical, 6-chromatic, claw-free graph is K_6. We prove the following main results in this paper. Theorem 1.4 is a generalization of a result obtained in [8] that no two vertices of degree $t + 1$ are adjacent in any non-complete, double-critical, t-chromatic graph.

Theorem 1.4. *If G is a non-complete, double-critical, t-chromatic graph with $t \geq 6$, then for any vertex $x \in V(G)$ with $d_G(x) = t + 1$, the following hold:*

(a) *$e(\overline{G[N_G(x)]}) \geq 8$; and*
(b) *for any vertex $y \in N_G(x)$, $d_G(y) \geq t + 4$. Furthermore, if $d_G(y) = t + 4$, then $|N_G(x) \cap N_G(y)| = t - 2$ and $\overline{G[N_G(x)]}$ contains either only one cycle, which is isomorphic to C_8, or exactly two cycles, each of which is isomorphic to C_5.*

Corollary 1.5 follows immediately from Theorem 1.4.

Corollary 1.5. *If G is a non-complete, double-critical, t-chromatic graph with $t \geq 6$, then no vertex of degree $t + 1$ is adjacent to a vertex of degree $t + 1$, $t + 2$, or $t + 3$ in G.*

We then use Corollary 1.5 to prove the following main result.

Theorem 1.6. *Let G be a double-critical, t-chromatic graph with $t \in \{6, 7, 8\}$. If G is claw-free, then $G = K_t$.*

The rest of this paper is organized as follows. In Section 2, we first list some known properties of non-complete, double-critical graphs obtained in [8] and then establish a few new ones. In particular, Lemma 2.4 turns out to be very useful. Our new lemmas lead to a very short proof of Theorem 1.6 for $t = 6, 7$, which we place at the end of Section 2. We prove the remainder of our main results in Section 3.

2. Preliminaries

The following is a summary of the basic properties of non-complete, double-critical graphs shown by Kawarabayashi, Pedersen, and Toft in [8].

Proposition 2.1. *If G is a non-complete, double-critical, t-chromatic graph, then all of the following are true.*

(a) *G does not contain K_{t-1} as a subgraph.*
(b) *For all edges xy, every $(t-2)$-coloring $c : V(G) \setminus \{x, y\} \to \{1, 2, \dots, t-2\}$ of $G \setminus \{x, y\}$, and any non-empty sequence $j_1, j_2, \dots, j_i$ of i different colors from $\{1, 2, \dots, t-2\}$, there is a path of order $i + 2$ with vertices $x, v_1, v_2, \dots, v_i, y$ in order such that $c(v_k) = j_k$ for all $k \in \{1, 2, \dots, i\}$.*