Discrete Mathematics 340 (2017) 1639–1644

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A Fan-type heavy triple of subgraphs for pancyclicity of 2-connected graphs

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A R T I C L E I N F O

Article history:
Received 16 April 2015
Received in revised form 30 September 2016
Accepted 11 March 2017
Available online 30 March 2017

Keywords:
Fan-type heavy subgraph
Hamilton cycle
Pancyclicity

A B S T R A C T

Graph G of order n is said to be pancyclic if it contains cycles of all lengths k for $k \in \{3, \dots, n\}$. A vertex $v \in V(G)$ is called super-heavy if its degree in G is at least $(n+1)/2$. For a given graph S we say that G is S-f_1-heavy if for every induced subgraph K of G isomorphic to S and every two vertices $u, v \in V(K)$, $d_K(u, v) = 2$ implies that at least one of them is super-heavy. For a family of graphs $\mathcal{S}$ we say that G is $\mathcal{S}$-f_1-heavy, if G is S-f_1-heavy for every graph $S \in \mathcal{S}$.

Let H denote the hourglass, a graph consisting of two triangles that have exactly one vertex in common. In this paper we prove that every 2-connected $\{K_{1,3}, P_7, H\}$-f_1-heavy graph on at least nine vertices is pancyclic or missing only one cycle. This result extends the previous work by Faudree, Ryjáček and Schiermeyer.

1. Introduction

We use [5] for terminology and notation not defined here. In the paper only finite, simple and undirected graphs are considered.

Let G be a graph on n vertices. G is said to be Hamiltonian, if it contains a cycle C_n, and it is called pancyclic, if it contains cycles of all lengths k for $3 \leq k \leq n$. If G does not contain an induced copy of a given graph S, we say that G is S-free. G is called S-f_i-heavy, if for every induced subgraph K of G isomorphic to S and for every two vertices $x, y \in V(K)$ satisfying $d_K(x, y) = 2$, the following inequality holds: $\max\{d_G(x), d_G(y)\} \geq (n+i)/2$. For the sake of simplicity, we write f-heavy instead of f_0-heavy. For a family of graphs $\mathcal{S}$ we say that G is $\mathcal{S}$-free ($\mathcal{S}$-f_i-heavy), if G is S-free (S-f_i-heavy, respectively) for every graph $S \in \mathcal{S}$. The complete bipartite graph $K_{1,3}$ is called a claw.

The notion of f-heaviness was introduced in [12]. It was inspired by the following well-known theorem.

Theorem 1 (Fan [6]). *Let G be a 2-connected graph of order $n \geq 3$. If*

$$d_G(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq n/2$$

for every pair of vertices u and v in G, then G is Hamiltonian.

Note that an equivalent formulation of Theorem 1 is that every 2-connected, P_3-f-heavy graph of order $n \geq 3$ is Hamiltonian. Clearly, for a given graph S every S-free graph is S-f_i-heavy for every integer i. Hence, it follows from Theorem 1 that every P_3-free graph is Hamiltonian (which is not surprising, since the only 2-connected P_3-free graph is a complete graph). Fan's result was extended in 1987 by Wojda and Benhocine (graph F_{4r} appearing in the following theorem consists of a clique on $2r$ vertices that is connected via a perfect matching with r disjoint copies of a path P_2).

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http://dx.doi.org/10.1016/j.disc.2017.03.010
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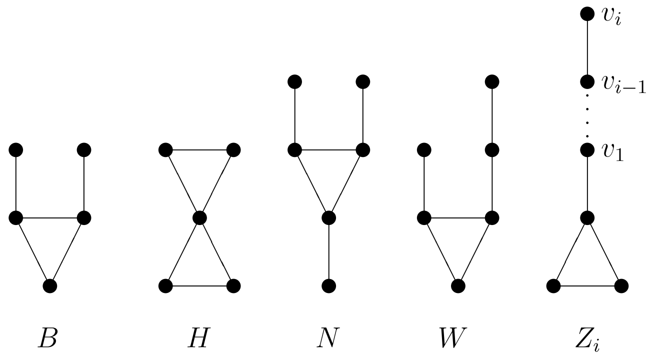

Fig. 1. Graphs B (Bull), H (Hourglass), N (Net), W (Wounded) and Z_i.

Theorem 2 (*Benhocine and Wojda [3]*). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is P_3-f-heavy, then G is pancyclic unless $n = 4r$, $r \geq 1$, and G is F_{4r} or else $n \geq 6$ is even and $G = K_{n/2,\,n/2}$ or $G = K_{n/2,\,n/2} - e$.*

Since none of the special graphs mentioned in Theorem 2 is P_3-f_1-heavy, it is easy to see that every 2-connected P_3-f_1-heavy graph is pancyclic. It is also not difficult to see that P_3 is the only connected graph S such that every 2-connected S-f_1-heavy graph is pancyclic (for details see [7, Theorem 13]). Bedrossian in his Ph.D. Thesis [1] considered pairs of graphs and managed to characterize all pairs of forbidden subgraphs implying Hamiltonicity and pancyclicity of 2-connected graphs. The fact that there are indeed no other pairs of graphs forbidding of which ensures Hamiltonicity or pancyclicity was showed a few years later by Faudree and Gould. The graphs B, N, W and Z_i involved in the following theorems are represented on Fig. 1.

Theorem 3 (*Bedrossian [1]; Faudree and Gould [8]*). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

Theorem 4 (*Bedrossian [1]; Faudree and Gould [8]*). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

Later both of these results were improved by numerous authors. Note that since every P_3-f-heavy graphs is also S-f-heavy for any connected graph S other than complete graph, Theorem 1 is a corollary from Theorem 5.

Theorem 5 (*Ning and Zhang [12]*). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-f-heavy implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

Theorem 6. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-f_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*
– Z_1 (Bedrossian, Chen and Schelp [2]),
- Z_2, P_4 (Ning [11]),
- P_5 (Wideł [14]).

Triples of forbidden subgraphs with respect to Hamiltonian properties have also been extensively examined. One of the many results obtained in this field is the following theorem (see Fig. 1 for the graph H).

Theorem 7 (*Faudree et al., Theorem 15 in [7]*). *Every 2-connected $\{K_{1,3}, P_7, H\}$-free graph on $n \geq 9$ vertices is pancyclic or missing only one cycle.*

Recently, Ning proved the following fact.

Theorem 8 (*Ning, [10]*). *Every 2-connected $\{K_{1,3}, P_7, H\}$-f-heavy graph is Hamiltonian.*

Motivated by Theorems 7 and 8 and by similar results for pairs of forbidden and Fan-type heavy subgraphs, in this paper we prove the following theorem.

Theorem 9. *Let G be a 2-connected, $\{K_{1,3}, P_7, H\}$-f_1-heavy graph. If there is a super-heavy vertex in G, then G is pancyclic.*